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SOA Exam MLC

Study Manual

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TO OUR READERS:

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Preface

Welcome to Exam MLC!

Exam MLC is the exam in which life actuaries learn how to price and reserve for an insurance whose benefits may not be paid for a long time. This requires dealing with both probabilities of events and interest—the topics of exams P and FM.

Syllabus

The Fall 2017 syllabus is posted at the following url:

`https://www.soa.org/Files/Edu/2017/fall/edu-2017-fall-exam-mlc-syllabus.pdf`

The topics are

1. Survival models
2. Insurances
3. Annuities
4. Premiums
5. Reserves
6. Markov chains
7. Multiple decrement models
8. Multiple life models
9. Pensions
10. Interest rate risk
11. Profit tests, participating insurance, and universal life

The textbook for the course is *Actuarial Mathematics for Life Contingent Risks* 2nd edition. This is a college-style textbook. It is oriented towards practical application rather than exam preparation, with an emphasis on non-U.S. practice. Almost all exercises require use of spreadsheets or derivation of formulas, not the typical sort of question you'd get on an exam (although written answer questions may ask you to derive formulas). In addition to the syllabus, you should read the introductory study note, found at

`www.soa.org/Files/Edu/2017/fall/2017-fall-exam-mlc-intro-notes.pdf`

In paragraph 7, the note mentions that the Learning Outcomes, not the recommended text sections, comprise the syllabus and guide the exam committee when writing questions. Usually the text sections cover the learning outcomes, but in a few areas the text does not cover the learning outcomes directly. The readings corresponding to each lesson are listed at the beginning of the lesson, and occasionally it is mentioned that the lesson is not directly covered by the textbook.

Here is the distribution of questions by topic for the Spring 2012 through Fall 2013 exams, and the points per topic for the 2014 and later exams. Note that each question is classified based on the highest lesson required for it, so a question involving an asset share on universal life (there was one such question) would be classified as a universal life question. Thus a 0 does not indicate no questions on the topic on the exam. For example, the Fall 2016 exam had a question involving interest rate models applied to a multiple-life insurance, even though the table shows 0 for multiple life models.

		Questions				Points						
Topic	Lessons	Spr 12	Fall 12	Spr 13	Fall 13	Spr 14	Fall 14	Spr 15	Fall 15	Spr 16	Fall 16	Spr 17
Survival distributions	2–8	3	2	1	3	4	4	2	11	2	6	2
Insurances	10–15	0	2	2	1	3	2	6	4	13	6	13
Annuities	17–22	4	2	1	3	2	5	0	0	2	0	2
Premiums	24–32	5	2	3	6	17	18	20	10	22	17	17
Reserves	34–41	3	5	5	4	14	8	24	4	6	10	2
Markov chains	43–45	3	3	3	3	9	12	15	6	16	7	14
Multiple decrement models	46–51	2	1	2	0	2	2	6	8	0	9	4
Multiple life models	53–59	2	2	2	1	7	8	2	11	2	0	9
Pensions	61	1	0	1	1	11	11	8	6	15	14	13
Interest rate models	62–63	2	2	2	1	7	0	0	10	0	11	0
Profit tests	64–68	5	4	3	2	20	26	13	26	18	16	20
Total		30	25	25	25	96	96	96	96	96	96	96

As you can see in this table, weights on the topics have varied. As mentioned in the next paragraph, the syllabus weight on pensions has increased twice within the last few years.

Changes to the syllabus

Pension plan funding, the material covered in Section 61.4, was added to the Spring 2016 syllabus. The weight on pensions is increased from 5–15% to 10–20%. The weight on pension was previously increased in Spring 2014.

Other downloads from the SOA site

The end of the introductory study note has links to various useful downloads.

Tables

Download the tables you will be given on the exam. They will often be needed for the exercises. They are the second link at the end of the introductory study note. The direct URL is

`http://www.soa.org/Files/Edu/edu-2013-mlc-tables.pdf`

The tables include the Illustrative Life Table, the Illustrative Service Table, some interest functions, and the standard normal distribution function.

The SOA has specified rules for using the normal distribution table they supply: *Do not interpolate in the table. Simply use the nearest value.* If you are looking for $\Phi(0.0244)$, use $\Phi(0.02)$. If you are given the cumulative probability $\Phi(x) = 0.8860$ and need x, use 1.21, the nearest x available.

Notation and terminology note

The note found at

`www.soa.org/Files/Edu/2016/fall/edu-2016-fall-exam-mlc-notation.pdf` discusses the terminology used on the exam. Often different textbooks use different names for the same concept. In almost all cases, the exam uses the terminology of *Actuarial Mathematics for Life Contingent Risks*. This manual uses the terminology that will be used on the exam.

Sample questions

The last three links at the end of the introductory study note provide 322 sample multiple choice question, their solutions, and 22 sample written answer questions and solutions. Most of the first 283 sample multiple choice questions are from old exams, but with terminology updated to the terminology of the current syllabus. The emphasis of exams has changed, and many of the concepts which are repeatedly used in those sample questions rarely appear on current exams. Sometimes in this manual, it is stated that the material is very helpful for answering the sample questions but not too important for current exams.

My version of the solutions to the sample multiple choice questions can be found in the manual. Use Tables C.9–C.10 to track them down.

This manual does not provide solutions to the sample written answer questions, since the official solution is the best guide as to what they expect. However, it does provide solutions to written answer questions from old exams. It may not be appropriate to use a shortcut to solve a written answer question if this shortcut is not in the textbook.

Old exam questions in this manual

There are about 700 original exercises in the manual and 1000 old exam questions. The old exam questions come from old Part 4, Part 4A, Course 150, Course 151 exams, 2000-syllabus Exam 3, Exam M, and Exam MLC. However, questions from the 2012 and later exams are not given in the exercises, so you may use those exams as practice exams.

SOA Part 4 in 1986 had morning and afternoon sessions. I indicate afternoon session questions with "A". The morning session had the more basic topics (through reserves), while the afternoon session had advanced topics (multiple lives, multiple decrements, etc.) Both sessions were multiple choice questions.

SOA Course 150 from 1987 through 1991 had multiple choice questions in the morning and written answer questions in the afternoon. Since MLC will include written answer questions in 2014, I've included all applicable written answer questions in the exercises.

The CAS Part 4A exams awarded varying numbers of points to questions; some are 1 point and some are 2 points. The 1 point questions are probably too easy for a modern exam, but they'll give you a little practice. The pre-1987 exams probably were still based on Jordan (the old textbook), but the questions I provided, while ancient, still have value. Similarly, the cluster questions on SOA Course 150 in the 1990s generally were awarded 1 point per question.

The first edition of *Actuarial Mathematics*, which was used until around 1997, had commutation functions. Some old exam questions from the period before 1997 used commutation functions. In some cases, I've adapted such questions for use without commutation functions. You will see some "based on" questions where I made this adaptation. Even though these questions still have a commutation function feel, they are still legitimate questions.

Although the CAS questions are limited to certain topics, are different stylistically, and are easier, they are a good starting point.

Course 151 is the least relevant to this subject. I've only included a small number of questions from 151 in the first lesson, which is background.

Back in 1999, the CAS and SOA created a sample exam for the then-new 2000 syllabus. This exam had some questions from previous exams but also some new questions, some of them not multiple choice. This sample exam was never a real exam, and some of its questions were defective. This sample exam is no longer available on the web. I have included appropriate questions from it. *Whenever an exercise is labeled C3 Sample, it refers to the 1999 sample, not the current list of 322 sample questions.*

Questions from CAS exams given in 2005 and later are not included in this manual. However, a list of relevant questions from those CAS exams appears at the end of each lesson if there are any relevant questions. Worked-out solutions to all Life Contingencies questions from those CAS exams are in Appendix B. More than half of the CAS exam questions on Life Contingencies are on single-life, multiple-life, and multiple-decrement probabilities, so they will not give you a balanced review of all topics covered by MLC.

Questions from old exams are marked xxx:yy, where xxx is the time the exam was given, with S for spring and F for fall followed by a 2-digit year, and yy is the question number. Sometimes xxx is preceded with SOA or CAS to indicate the sponsoring organization. From about 1986 to 2000, SOA exams had 3-digit numbers (like 150) and CAS exams were a number and a letter (like 4A). From 2000 to Spring 2003, the exams were jointly sponsored. There was a period in the 1990s when the SOA, while it allowed use of its old exam questions, did not want people to reveal which exam they came from. As a result, I sometimes had study notes for old exams in this period and could not identify the exam they came from. In such a case, I mark the question aaa-bb-cc:yy, where aaa-bb-cc is the study note number and yy is the question number. Generally aaa is the exam number (like 150), and cc is the 2-digit year the study note was published.

Characteristics of this exam

The exam will have 20 multiple choice questions worth 2 points apiece and 6–7 written answer questions totalling 56 points, for a total of 96 points. You will be given 4 hours to complete the exam, or 2.5 minutes per point. The multiple choice questions will be easy, and are meant to screen students. Only students who perform above a threshold score on the multiple choice part of the exam will have their written answer questions graded.

For multiple choice questions, there is no penalty for guessing. Fill in all questions regardless of whether you have time to work out the question or not—you lose nothing and you may be lucky!

The answer choices on SOA exams are almost always specific answers, not ranges.

For written answer questions, you will be graded on your work as well as your final answer. This manual has many shortcuts, but to the extent they are not in the textbook, you may have to derive the shortcut before using it. Written answer questions will be in many parts. Although most of the points will involve calculation, some parts of the question may ask you to explain concept or to determine the effect of varying a parameter. Sometimes a written answer question may ask you to derive a formula. This manual provides derivations for most formulas; make sure you understand these derivations.

The textbook occasionally provides background non-mathematical information. For example, it discusses various life insurance products. It discusses pros and cons of dividends paid in cash versus dividends reinvested in reversionary bonuses. I've placed some of this material in Lesson 69.

New for the 15^{th} edition

The added syllabus material is included in the pension lesson.

The written answer questions on the practice exams have been revamped to be similar to the real exam in size. The practice exams were reweighted to put heavier weight on pension, in accordance with the syllabus change. Unused questions were moved to a new supplementary questions lesson. You can practice with those questions before trying the practice exams.

Solutions to all questions from released exams through Spring 2017 have been added.

Acknowledgements

I would like to thank the SOA and CAS for allowing me to use their old exam questions. I'd also like to thank Harold Cherry for suggesting this manual and for providing three of the pre-2000 SOA exams and all of the pre-2000 CAS exams I used.

The creators of TEX, LATEX, and its multitude of packages all deserve thanks for making possible the professional typesetting of this mathematical material.

Kevin Blackman deserves special thanks for checking the solutions to most of the exercises in the first edition of the manual.

Many readers have provided errata lists; their contributions are appreciated, whether they consisted of multiple e-mails with large errata lists or simply pointing out a single typo. Particularly notable contributions were provided by

> Don Neville, who acted virtually as a copy editor; about 90% of the errata for the first edition came from him.
>
> Yitzy Lowy, who similarly provided the vast majority of the Life Contingency errors of the third edition, as well as useful shortcuts I was not aware of, and which are now incorporated.
>
> Rick Sutherland, who provided about 20 errata lists.
>
> Simon Schurr, who corrected both major and minor errors of the eighth edition, as well as offering numerous improved solutions to the exercises and practice exam questions.

A partial list of other readers who submitted errata is: Kirsten Aagesen, Victor Alvarez, Jason Aizkalns, Chris Allison, Tophe Anderson, Carter Angell, Elissa Aoun, John Alexander Arthur, Rich Astudillo, Jeff

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Errata

Please report all errors you find in these notes to the author. You may send them to the publisher at `mail@studymanuals.com` or directly to me at `errata@aceyourexams.net`. Please identify the manual and edition the error is in. This is the 15th edition 4th printing of the SOA Exam MLC manual.

An errata list will be posted at `errata.aceyourexams.net`. Check this errata list frequently.

Flashcards

Many students find flashcards a useful tool for learning key formulas and concepts. ASM flashcards, available from the same distributors that sell this manual, contain the formulas and concepts from this manual in a convenient deck of cards. The cards have crossreferences, usually by page, to the manual.

Lesson 1

Probability Review

This lesson is a brief summary of probability concepts you will need in the course. You will not be directly tested on these topics, but they are essential background. If you find this review too brief, you should review your favorite probability textbook for more details. Conversely, you may skip this lesson if you are familiar with the concepts.

1.1 Functions and moments

The *cumulative distribution function* $F(x)$ of a random variable X, usually just called the *distribution function*, is the probability that X is less than or equal to x:

$$F(x) = \Pr(X \le x)$$

It defines X, and is right-continuous, meaning $\lim_{h\to 0} F(x+h) = F(x)$ for h positive.

Some random variables are discrete (there are isolated points x at which $\Pr(X = x)$ is nonzero) and some are continuous (meaning $F(x)$ is continuous, and differentiable except at a countable number of points). Some are mixed—they are continuous except at a countable number of points.

Here are some important functions that are related to $F(x)$:

- $S(x)$ is the survival function, the complement of $F(x)$, the probability that X is strictly greater than x.

  $$S(x) = \Pr(X > x)$$

  It is called the survival function since if X represents survival time, it is the probability of surviving longer than x.

- For a continuous random variable, $f(x) = \frac{\mathrm{d}}{\mathrm{d}x}F(x)$ is the probability density function. For a discrete random variable, the probability mass function $f(x) = \Pr(X = x)$ serves a similar purpose.

- For a continuous random variable, $\lambda(x) = \dfrac{f(x)}{S(x)} = -\dfrac{\mathrm{d}\ln S(x)}{\mathrm{d}x}$ is the hazard rate function.[1] Sometimes the hazard rate function is denoted by $h(x)$ instead of $\lambda(x)$. The hazard rate function is like a conditional density function, the conditional density given survival to time x. We can reverse the operations to go from $\lambda(x)$ to $S(x)$: $S(x) = \exp\left(-\int_{-\infty}^{x} \lambda(u)\mathrm{d}u\right)$.

Why do we bother differentiating F to obtain f? Because the density is needed for calculating moments. Moments of a random variable measure its center and dispersion. The *expected value* of X is defined by

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f(x)\mathrm{d}x$$

and more generally the expected value of a function of a random variable is defined by

$$\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\mathrm{d}x$$

[1]As we'll learn in Lesson 3, in International Actuarial Notation (IAN), μ_x is used for the hazard rate function.

For discrete variables, the integrals are replaced with sums.

The n^{th} raw moment of X is defined as $\mu'_n = \mathbf{E}[X^n]$. $\mu = \mu'_1$ is the *mean*. The n^{th} central moment of X ($n \neq 1$) is defined as[2] $\mu_n = \mathbf{E}[(X-\mu)^n]$. Usually n is a positive integer, but it need not be. When we mention moments in this manual and don't state otherwise, we mean *raw* moments.

Expectation is linear, so the central moments can be calculated from the raw moments by binomial expansion. In the binomial expansion, the last two terms always merge, so we have

$$\mu_2 = \mu'_2 - \mu^2 \qquad \text{instead of } \mu'_2 - 2\mu'_1\mu + \mu^2 \tag{1.1}$$
$$\mu_3 = \mu'_3 - 3\mu'_2\mu + 2\mu^3 \qquad \text{instead of } \mu'_3 - 3\mu'_2\mu + 3\mu'_1\mu^2 - \mu^3 \tag{1.2}$$
$$\mu_4 = \mu'_4 - 4\mu'_3\mu + 6\mu'_2\mu^2 - 3\mu^4 \qquad \text{instead of } \mu'_4 - 4\mu'_3\mu + 6\mu'_2\mu^2 - 4\mu'_1\mu^3 + \mu^4$$

Special functions of moments are:

- The *variance* is $\text{Var}(X) = \mu_2$, and is denoted by σ^2.
- The *standard deviation* σ is the positive square root of the variance.
- The *coefficient of variation* is σ/μ. This concept appears in the *Actuarial Mathematics for Life Contingent Risks* 2nd edition exercise 11.4, but otherwise does not appear in this course. However, it plays a big role in Exam C.

We will discuss important things you should know about variance in Section 1.3. For the meantime, I will repeat formula (1.1) using different notation, since it's so important:

$$\text{Var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2 \tag{1.3}$$

Many times this is the best way to calculate variance.

For two random variables X and Y:

- The *covariance* is defined by $\text{Cov}(X,Y) = \mathbf{E}[(X-\mu_X)(Y-\mu_Y)]$.
- The *correlation coefficient* is defined by $\rho_{XY} = \frac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y}$.

As with the variance, another formula for covariance is

$$\text{Cov}(X,Y) = \mathbf{E}[XY] - \mathbf{E}[X]\,\mathbf{E}[Y]$$

Note that $\mathbf{E}[XY] \neq \mathbf{E}[X]\,\mathbf{E}[Y]$ in general. In fact, $\mathbf{E}[XY] = \mathbf{E}[X]\,\mathbf{E}[Y]$ if and only if X and Y are uncorrelated, in other words if their correlation is 0.

For independent random variables, $\text{Cov}(X,Y) = 0$.

A *$100p^{th}$ percentile* is a number π_p such that $F(\pi_p) \geq p$ and $F(\pi_p^-) \leq p$. If F is continuous and strictly increasing, it is the unique point at which $F(\pi_p) = p$. In this course, we will only discuss percentiles for strictly increasing distribution functions, and that will simplify matters. A *median* is a 50^{th} percentile.

A *mode* is x such that $f(x)$ (or $\Pr(X = x)$ for a discrete distribution) is maximized.

1.2 Probability distributions

A couple of probability distributions will be used frequently during this course.

[2] This μ_n has no connection to μ_x, the force of mortality.

1.2.1 Bernoulli distribution

A random variable has a Bernoulli distribution if it only assumes the values of 0 and 1. The value of 1 is assumed with probability q.

If X is Bernoulli, then its mean is q—the same as the probability of 1. Its variance is $q(1-q)$.

We can generalize the variable. Consider a random variable Y which can assume only two values, but the two values are x_1 and x_2 instead of 0 or 1; the probability of x_2 is q. Then $Y = x_1 + (x_2 - x_1)X$, where X is Bernoulli. It follows that the mean is $x_1 + (x_2 - x_1)q$. More importantly, the variance is $(x_2 - x_1)^2 q(1-q)$. This is a fast way to calculate variance, faster than calculating $\mathbf{E}[Y]$ and $\mathbf{E}[Y^2]$, so remember it. To repeat:

To compute the variance of a Bernoulli-type variable assuming only two values, multiply the product of the probabilities of the two values by the square of the difference between the two values.

I call this trick for calculating the variance the *Bernoulli shortcut*.

Example 1A For a one-year term life insurance policy of 1000:

(i) The premium is 30.
(ii) The probability of death during the year is 0.02.
(iii) The company has expenses of 2.
(iv) If the insured survives to the end of the year, the company pays a dividend of 3.

Ignore interest.

Calculate the variance in the amount of profit the company makes on this policy.

Answer: There are only two possibilities—either the insured dies or he doesn't—so we have a Bernoulli here. We can ignore premium and expenses, since they don't vary, so they generate no variance. Either the company pays 1000 (probability 0.02) or it pays 3 (probability 0.98). The variance is therefore

$$(1000-3)^2(0.02)(0.98) = \boxed{\mathbf{19{,}482.5764}}.$$

□

Quiz 1-1 [3] A random variable X has the following distribution:

x	$\Pr(X = x)$
20	0.7
120	0.3

Calculate $\text{Var}(X)$.

A sum of m Bernoulli random variables each with the same mean q is a *binomial random variable*. Its mean is mq and its variance is $mq(1-q)$.

1.2.2 Uniform distribution

The uniform distribution on $[a,b]$ is a continuous distribution with constant density $1/(b-a)$ on the interval $[a,b]$ and 0 elsewhere. Its mean is its midpoint, $(a+b)/2$, and its variance is $(b-a)^2/12$.

It is a simple distribution, and will be used heavily in examples throughout the course.

[3]Quiz solutions are at the end of the lesson, after exercise solutions.

1.2.3 Exponential distribution

The exponential distribution is defined by cumulative distribution function $F(x) = 1-e^{-x/\theta}$, where θ is the mean. The density function is $f(x) = e^{-x/\theta}/\theta$. This density function—an exponentiated variable—is very convenient to use in conjunction with other exponentiated items, such as those that arise from compound interest. Therefore, this distribution will be used heavily in examples throughout the course.

The sum of n independent exponential random variables all having the same mean is a gamma random variable. If Y is gamma and is the sum of n exponential random variables with mean θ, then its density function is

$$f_Y(x) = \frac{x^{n-1}e^{-x/\theta}}{\Gamma(n)\theta^n}$$

where $\Gamma(n)$, the gamma function, is a continuous extension of the factorial function; for n an integer, $\Gamma(n) = (n-1)!$ By using $\Gamma(n)$ instead of $(n-1)!$, the gamma function can be defined for non-integral n.

1.3 Variance

Expected value is linear, meaning that $\mathbf{E}[aX + bY] = a\,\mathbf{E}[X] + b\,\mathbf{E}[Y]$, regardless of whether X and Y are independent or not. Thus $\mathbf{E}[(X+Y)^2] = \mathbf{E}[X^2]+2\,\mathbf{E}[XY]+\mathbf{E}[Y^2]$, for example. This means that $\mathbf{E}[(X+Y)^2]$ is *not* equal to $\mathbf{E}[X^2] + \mathbf{E}[Y^2]$ (unless $\mathbf{E}[XY] = 0$).

Also, it is *not* true in general that $\mathbf{E}\big[g(X)\big] = g\big(\mathbf{E}[X]\big)$. So $\mathbf{E}[X^2] \neq \big(\mathbf{E}[X]\big)^2$.

Since variance is defined in terms of expected value, $\mathrm{Var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2$, this allows us to develop a formula for $\mathrm{Var}(aX + bY)$. If you work it out, you get

$$\mathrm{Var}(aX + bY) = a^2\,\mathrm{Var}(X) + 2ab\,\mathrm{Cov}(X, Y) + b^2\,\mathrm{Var}(Y) \tag{1.4}$$

In particular, if $\mathrm{Cov}(X, Y) = 0$ (which is true if X and Y are independent), then

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

and generalizing to n independent variables,

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n}\mathrm{Var}(X_i)$$

If all the X_i's are independent and have identical distributions, and we set $X = X_i$ for all i, then

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = n\,\mathrm{Var}(X) \tag{1.5}$$

However, $\mathrm{Var}(nX) = n^2\,\mathrm{Var}(X)$, not $n\,\mathrm{Var}(X)$. You must distinguish between these two situations, which are quite different. Think of the following example. The stock market goes up or down randomly each day. We will assume that each day's change is independent of the previous day's, and has the same distribution. Compare the variance of the following possibilities:

1. You put \$1 in the market, and leave it there for 10 days.

2. You put \$10 in the market, and leave it there for 1 day.

In the first case, there are going to be potential ups and downs each day, and the variance of the change of your investment will be 10 times the variance of one day's change because of this averaging. In the second case, however, you are multiplying the variation of a single day's change by 10—there's no dampening of the change by 10 different independent random events, the change depends on a single random event. As a result, you are multiplying the variance of a single day's change by 100.

In the more general case where the variables are not independent, you need to know the covariance. This can be provided in a *covariance matrix*. If you have n random variables $X_1, \ldots, X_n$, this $n \times n$ matrix $\mathbf{A}$ has $a_{ij} = \mathrm{Cov}(X_i, X_j)$ for $i \neq j$. For $i = j$, $a_{ii} = \mathrm{Var}(X_i)$. This matrix is symmetric and non-negative definite. However, the covariance of two random variables may be negative.

Example 1B For a loss X on an insurance policy, let X_1 be the loss amount and X_2 the loss adjustment expenses, so that $X = X_1 + X_2$. The covariance matrix for these random variables is

$$\begin{pmatrix} 25 & 5 \\ 5 & 2 \end{pmatrix}$$

Calculate the variance in total cost of a loss including loss adjustment expenses.

Answer: In formula (1.4), $a = b = 1$. From the matrix, $\mathrm{Var}(X_1) = 25$, $\mathrm{Cov}(X_1, X_2) = 5$, and $\mathrm{Var}(X_2) = 2$. So

$$\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + 2\,\mathrm{Cov}(X_1, X_2) + \mathrm{Var}(X_2) = 25 + 2(5) + 2 = \boxed{37}$$ □

A sample is a set of observations from n independent identically distributed random variables. The sample mean $\bar{X}$ is the sum of the observations divided by n. The variance of the sample mean of $X_1, \ldots, X_n$, which are observations from the random variable X, is

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{\sum_{i=1}^n X_i}{n}\right) = \frac{n\,\mathrm{Var}(X)}{n^2} = \frac{\mathrm{Var}(X)}{n} \tag{1.6}$$

1.4 Normal approximation

The Central Limit Theorem says that for any distribution with finite variance, the sample mean of a set of independent identically distributed random variables approaches a normal distribution. By the previous section, the mean of the sample mean of observations of X is $\mathbf{E}[X]$ and the variance is σ^2/n. These parameters uniquely determine the normal distribution that the sample mean converges to. A random variable Y with normal distribution with mean μ and variance σ^2 can be expressed in terms of a standard normal random variable Z in the following way:

$$Y = \mu + \sigma Z$$

and you can look up the distribution of Z in a table of the standard normal distribution function that you get at the exam.

The normal approximation of a percentile of a random variable is performed by finding the corresponding percentile of a normal distribution with the same mean and variance. Let $\Phi(x)$ be the cumulative distribution function of the standard normal distribution. (The standard normal distribution has $\mu = 0$, $\sigma = 1$. Φ is the symbol generally used for this distribution function.) Suppose we are given that X is a normal random variable with mean μ, variance σ^2; we will write $X \sim n(\mu, \sigma^2)$ to describe X. And suppose we want to calculate the 95$^{\text{th}}$ percentile of X; in other words, we want a number x such that

$\Pr(X \le x) = 0.95$. We would reason as follows:

$$\begin{aligned}\Pr(X \le x) &= 0.95\\ \Pr\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) &= 0.95\\ \Phi\left(\frac{x-\mu}{\sigma}\right) &= 0.95\\ \frac{x-\mu}{\sigma} &= \Phi^{-1}(0.95)\\ x &= \mu + \sigma\Phi^{-1}(0.95)\end{aligned}$$

Note that $\Phi^{-1}(0.95) = 1.645$ is a commonly used percentile of the normal distribution, and is listed at the bottom of the table you get at the exam.

You should internalize the above reasoning so you don't have to write it out each time. Namely, to calculate a percentile of a random variable being approximated normally, find the value of x such that $\Phi(x)$ is that percentile. Then *scale* x: multiply by the standard deviation, and then *translate* x: add the mean.

This approximation will be used repeatedly throughout the course.

Example 1C A big fire destroyed a building in which 100 of your insureds live. Each insured has a fire insurance policy. The losses on this policy follow a distribution with mean 1000 and variance 3,000,000. Even though all the insureds live in the same building, the losses are independent. You are now setting up a reserve for the cost of these losses.

Using the normal approximation, calculate the size of the reserve you should put up if you want to have a 95% probability of having enough money in the reserve to pay all the claims.

Answer: The expected total loss is the sum of the means, or (100)(1000) = 100,000. The variance of the total loss is the sum of the variances, or $100(3{,}000{,}000) = 3\times 10^8$. The standard deviation $\sigma = \sqrt{3\times 10^8} = 17{,}320.51$. For a standard normal distribution, the 95th percentile is 1.645. We scale this by 17,320.51 and translate it by 100,000: $100{,}000 + 17{,}320.51(1.645) = \boxed{128{,}492.24}$. □

The normal approximation is also used for probabilities. To approximate the probability that a random variable is less than x, calculate the probability that a normal random variable with the same mean and variance is less than x. In other words, calculate $\Phi\big((x-\mu)/\sigma\big)$. In this course, however, it will be rare that we approximate probabilities.

Example 1D A big fire destroyed a building in which 100 of your insureds live. Each insured has a fire insurance policy. The losses on this policy follow a distribution with mean 1000 and variance 3,000,000. Even though all the insureds live in the same building, the losses are independent. You are now setting up a reserve for the cost of these losses.

Using the normal approximation, calculate the probability that the average loss is less than 1100.

Answer: The mean of the average is 1000 and the variance of the average is $3{,}000{,}000/100 = 30{,}000$, as we just mentioned in formula (1.6). Therefore

$$\Pr(\bar{X} < 1100) \approx \Phi\left(\frac{1100-1000}{\sqrt{30{,}000}}\right) = \Phi(0.577) = \boxed{0.7190}$$

where we've evaluated $\Phi(0.577)$ as $\Phi(0.58)$ from the printed normal distribution tables. □

1.5 Conditional probability and expectation

The probability of event A given B, assuming $\Pr(B) \neq 0$, is

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

where $\Pr(A \cap B)$ is the probability of both A and B occurring. A corresponding definition for continuous distributions uses the density function f instead of Pr:

$$f_X(x \mid y) = \frac{f(x, y)}{f(y)}$$

where $f(y) = \int f(x, y)\mathrm{d}x \neq 0$.

Two important theorems are *Bayes Theorem* and the *Law of Total Probability*:

Theorem 1.1 (Bayes Theorem)

$$\Pr(A \mid B) = \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)} \tag{1.7}$$

Correspondingly for continuous distributions

$$f_X(x \mid y) = \frac{f_Y(y \mid x) f_X(x)}{f_Y(y)} \tag{1.8}$$

Theorem 1.2 (Law of Total Probability) If B_i is a set of exhaustive (in other words, $\sum_i \Pr(B_i) = 1$) and mutually exclusive (in other words $\Pr(B_i \cap B_j) = 0$ for $i \neq j$) events, then for any event A,

$$\Pr(A) = \sum_i \Pr(A \cap B_i) = \sum_i \Pr(B_i)\Pr(A \mid B_i) \tag{1.9}$$

Correspondingly for continuous distributions,

$$\Pr(A) = \int \Pr(A \mid x) f(x)\mathrm{d}x \tag{1.10}$$

Expected values can be factored through conditions too. In other words, the mean of the means is the mean, or:

Conditional Mean Formula

$$\mathbf{E}_X[X] = \mathbf{E}_Y\big[\mathbf{E}_X[X \mid Y]\big] \tag{1.11}$$

This formula is one of the *double expectation* formulas. More generally for any function g

$$\mathbf{E}_X[g(X)] = \mathbf{E}_Y[\mathbf{E}_X[g(X) \mid Y]] \tag{1.12}$$

Here are examples of this important theorem. Versions of the first example appear very frequently on this exam.

Example 1E There are 2 types of actuarial students, bright and not-so-bright. The bright ones pass 80% of the exams they take and the not-so-bright ones pass 40% of the exams they take. All students start with Exam 1 and take the exams in sequence, and drop out as soon as they fail one exam. An equal number of bright and not-so-bright students take Exam 1.

Determine the probability that a randomly selected student taking Exam 3 will pass.

Answer: A common wrong answer to this question is $0.5(0.8) + 0.5(0.4) = 0.6$. This is an incorrect application of the Law of Total Probability. The probability that a student taking Exam 3 is bright is more than 0.5, because of the elimination of the earlier exams.

A correct way to calculate the probability is to first calculate the probability that a student is taking Exam 3 given the two types of students. Let I_1 be the event of being bright initially (before taking Exam 1) and I_2 the event of not being bright initially. Let E be the event of taking Exam 3. Then by Bayes Theorem and the Law of Total Probability,

$$\Pr(I_1 \mid E) = \frac{\Pr(E \mid I_1)\Pr(I_1)}{\Pr(E)}$$
$$\Pr(E) = \Pr(E \mid I_1)\Pr(I_1) + \Pr(E \mid I_2)\Pr(I_2)$$

Now, the probability that one takes Exam 3 if bright is the probability of passing the first two exams, or $0.8^2 = 0.64$. If not-so-bright, the probability is $0.4^2 = 0.16$. So we have

$$\Pr(E) = 0.64(0.5) + 0.16(0.5) = 0.4$$
$$\Pr(I_1 \mid E) = \frac{(0.64)(0.5)}{0.4} = 0.8$$

and $\Pr(I_2 \mid E) = 1 - 0.8 = 0.2$ (or you could go through the above derivation with I_2 instead of I_1). Now we're ready to apply the Law of Total Probability to the conditional distributions given E to answer the question. Let P be the event of passing Exam 3. Then

$$\Pr(P \mid E) = \Pr(P \mid I_1\&E)\Pr(I_1 \mid E) + \Pr(P \mid I_2\&E)\Pr(I_2 \mid E)$$
$$= (0.8)(0.8) + (0.4)(0.2) = \boxed{\mathbf{0.72}}$$

□

Now let's do a continuous example.

Example 1F Claim sizes follow an exponential distribution with mean θ. θ varies by insured. Over all insureds, θ has a distribution with the following density function:

$$f(\theta) = \frac{1}{\theta^2} \qquad 1 \le \theta < \infty$$

Calculate the probability that a claim from a randomly selected insured will be greater than 0.5.

Answer: The probability that a claim is greater than 0.5 *given an insured with claim sizes having an exponential distribution with mean θ* is

$$1 - F(0.5 \mid \theta) = 1 - \left(1 - e^{-0.5/\theta}\right) = e^{-0.5/\theta}$$

By the Law of Total Probability, the probability of a claim greater than 0.5 from a randomly selected insured is therefore

$$\Pr(X > 0.5) = \int_1^\infty e^{-0.5/\theta}\left(\frac{1}{\theta^2}\right)\mathrm{d}\theta$$
$$= 2e^{-0.5/\theta}\Big|_1^\infty$$
$$= 2\left(1 - e^{-0.5}\right) = 2(1 - 0.606531) = \boxed{\mathbf{0.786939}}$$

□

If $f(x \mid y) = f(x)$ for all x and y, we say that X and Y are *independent* random variables. Independent random variables are uncorrelated (but not conversely), so for X, Y independent, $\mathbf{E}[XY] = \mathbf{E}[X]\,\mathbf{E}[Y]$.

1.6 Conditional variance

Suppose we wish to calculate the variance of a random variable X. Rather than calculating it directly, it may be more convenient to condition X on Y, and then calculate moments of the conditional variable $X \mid Y$. Consider the following example:

Example 1G A life insurance agent may be happy or sad. The probability of happiness is 0.8. On a day when the agent is happy, the number of policies sold is exponentially distributed with mean 0.3. When the agent is sad, the number of policies sold is exponentially distributed with mean 0.1.

Calculate the variance of the number of policies sold per day.

Answer: One way to attack this problem is to calculate first and second moments and then variance. We condition X, the number of policies sold, on happiness, or Y. From equation (1.11) and the more general (1.12) with $g(X) = X^2$,

$$\mathbf{E}[X] = \mathbf{E}\big[\mathbf{E}[X \mid Y]\big] = \mathbf{E}[0.3, 0.1] = 0.8(0.3) + 0.2(0.1) = 0.26$$

The second moment of an exponential is the variance plus the mean squared, and the variance equals the mean squared, so the second moment of an exponential is twice the mean squared.

$$\mathbf{E}[X^2] = \mathbf{E}\big[\mathbf{E}[X^2 \mid Y]\big] = \mathbf{E}[2(0.3^2), 2(0.1)^2] = 0.8(0.18) + 0.2(0.02) = 0.148$$

So the variance is $\mathrm{Var}(X) = 0.148 - 0.26^2 = \boxed{\mathbf{0.0804}}$.

It is tempting, for those not in the know, to try to calculate the variance by weighting the two variances of happiness and sadness. In each state, the variance is the square of the mean, so the calculation would go $0.8(0.3^2) + 0.2(0.1^2) = 0.074$. But that is the wrong answer. It is too low. Do you see what is missing?

What is missing is the variance of the states. To capture the full variance, you must add the expected value of the variance of the states and the variance of the expected values of the states. The correct formula is

Conditional Variance Formula

$$\mathrm{Var}_X(X) = \mathbf{E}_Y[\mathrm{Var}_X(X \mid Y)] + \mathrm{Var}_Y(\mathbf{E}_X[X \mid Y]) \tag{1.13}$$

In our example, we've computed the expected value of the variances as 0.074. The state is a Bernoulli variable, and the expected values of the states are 0.3 and 0.1. So by the Bernoulli shortcut, the variance of the expected values is $(0.8)(0.2)(0.3 - 0.1)^2 = 0.0064$. The variance of the number of policies sold is $0.074 + 0.0064 = \boxed{\mathbf{0.0804}}$. □

Quiz 1-2 Given Y, the variable X has a normal distribution with mean Y and variance Y^2. Y is uniformly distributed on $[-10, 2]$.

Calculate the variance of X.

Exercises

Functions and moments

1.1. **[CAS3-F04:24]** A pharmaceutical company must decide how many experiments to run in order to maximize its profits.

- The company will receive a grant of $1 million if one or more of its experiments is successful.
- Each experiment costs $2,900.
- Each experiment has a 2% probability of success, independent of the other experiments.
- All experiments run simultaneously.
- Fixed expenses are $500,000.
- Ignore investment income.

The company performs the number of experiments that maximizes its expected profit.

Determine the company's expected profit before it starts the experiments.

(A) 77,818 (B) 77,829 (C) 77,840 (D) 77,851 (E) 77,862

Variance

1.2. **[4B-S93:9]** (1 point) If X and Y are independent random variables, which of the following statements are true?

1. $\operatorname{Var}(X+Y) = \operatorname{Var}(X) + \operatorname{Var}(Y)$
2. $\operatorname{Var}(X-Y) = \operatorname{Var}(X) + \operatorname{Var}(Y)$
3. $\operatorname{Var}(aX+bY) = a^2\,\mathbf{E}[X^2] - a(\mathbf{E}[X])^2 + b^2\,\mathbf{E}[Y^2] - b(\mathbf{E}[Y])^2$

(A) 1 (B) 1,2 (C) 1,3 (D) 2,3 (E) 1,2,3

1.3. **[4B-F95:28]** (2 points) Two numbers are drawn independently from a uniform distribution on [0,1].

What is the variance of their product?

(A) 1/144 (B) 3/144 (C) 4/144 (D) 7/144 (E) 9/144

Exercises continue on the next page . . .

Table 1.1: Important formulas from this lesson

$$\operatorname{Var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2 \tag{1.3}$$

$$\operatorname{Var}(aX + bY) = a^2 \operatorname{Var}(X) + 2ab \operatorname{Cov}(X, Y) + b^2 \operatorname{Var}(Y) \tag{1.4}$$

$$\operatorname{Var}(\bar{X}) = \frac{\operatorname{Var}(X)}{n} \tag{1.6}$$

$$\Pr(A \mid B) = \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)} \qquad \text{(Bayes Theorem—discrete)} \tag{1.7}$$

$$f_X(x \mid y) = \frac{f_Y(y \mid x) f_X(x)}{f_Y(y)} \qquad \text{(Bayes Theorem—continuous)} \tag{1.8}$$

$$\Pr(A) = \sum_i \Pr(A \cap B_i) = \sum_i \Pr(B_i)\Pr(A \mid B_i) \qquad \text{Law of Total Probability—discrete} \tag{1.9}$$

$$\Pr(A) = \int \Pr(A \mid x) f(x) \mathrm{d}x \qquad \text{Law of Total Probability—continuous} \tag{1.10}$$

$$\mathbf{E}_X[X] = \mathbf{E}_Y\big[\mathbf{E}_X[X \mid Y]\big] \qquad \text{(Double expectation)} \tag{1.11}$$

$$\operatorname{Var}_X(X) = \mathbf{E}_Y[\operatorname{Var}_X(X \mid Y)] + \operatorname{Var}_Y(E_X[X \mid Y]) \qquad \text{(Conditional variance)} \tag{1.13}$$

Distribution	Mean	Variance
Bernoulli	q	$q(1-q)$
Binomial	mq	$mq(1-q)$
Uniform on $[a, b]$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	θ	θ^2

Bernoulli shortcut: If a random variable can only assume two values a and b with probabilities q and $1-q$ respectively, then its variance is $q(1-q)(b-a)^2$.

1.4. **[151-82-92:4]** A company sells group travel-accident life insurance with b payable in the event of a covered individual's death in a travel accident.

The gross premium for a group is set equal to the expected value plus the standard deviation of the group's aggregate claims.

The standard premium is based on the following assumptions:

(i) All individual claims within the group are mutually independent; and
(ii) $b^2q(1-q) = 2500$, where q is the probability of death by travel accident for an individual.

In a certain group of 100 lives, the independence assumption fails because three specific individuals always travel together. If one dies in an accident, all three are assumed to die.

Determine the difference between this group's premium and the standard premium.

(A) 0 (B) 15 (C) 30 (D) 45 (E) 60

Exercises continue on the next page . . .

1.5. You are given the following information about the random variables X and Y:

(i) $\text{Var}(X) = 9$
(ii) $\text{Var}(Y) = 4$
(iii) $\text{Var}(2X - Y) = 22$

Determine the correlation coefficient of X and Y.

(A) 0 (B) 0.25 (C) 0.50 (D) 0.75 (E) 1

1.6. **[151-82-93:9]** (1 point) For a health insurance policy, trended claims will be equal to the product of the claims random variable X and a trend random variable Y.

You are given:

(i) $\mathbf{E}[X] = 10$
(ii) $\text{Var}(X) = 100$
(iii) $\mathbf{E}[Y] = 1.20$
(iv) $\text{Var}(Y) = 0.01$
(v) X and Y are independent

Determine the variance of trended claims.

(A) 144 (B) 145 (C) 146 (D) 147 (E) 148

1.7. X and Y are two independent exponentially distributed random variables. You are given that $\text{Var}(X) = 25$ and $\text{Var}(XY) = 7500$.

Determine $\text{Var}(Y)$.

(A) 25 (B) 50 (C) 100 (D) 200 (E) 300

Normal approximation

1.8. The number of policies a life insurance agent sells in one day is 1 with probability $1/5$ and 0 with probability $4/5$.

Assume the agent works 252 days a year.

Using the normal approximation, determine the 95^{th} percentile of the number of policies sold in one year.

1.9. A life insurance company has determined that the present value of profit on selling one contract is uniformly distributed on $[-50, 70]$.

Using the normal approximation, calculate the probability of making a profit on a portfolio of 50 policies.

Exercises continue on the next page . . .

Bernoulli shortcut

1.10. **[4B-F99:7]** (2 points) A player in a game may select one of two fair, six-sided dice. Die A has faces marked with 1, 2, 3, 4, 5 and 6. Die B has faces marked with 1, 1, 1, 6, 6, and 6. If the player selects Die A, the payoff is equal to the result of one roll of Die A. If the player selects Die B, the payoff is equal to the mean of the results of n rolls of Die B.

The player would like the variance of the payoff to be as small as possible.

Determine the smallest value of n for which the player should select Die B.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Conditional probability

1.11. **[M-F05:17]** The length of time, in years, that a person will remember an actuarial statistic is modeled by an exponential distribution with mean $1/Y$. In a certain population, the probability density function of Y is

$$f(y) = \frac{ye^{-y/2}}{4} \qquad y \geq 0$$

Calculate the probability that a person drawn at random from this population will remember an actuarial statistic less than $1/2$ year.

(A) 0.125 (B) 0.250 (C) 0.500 (D) 0.750 (E) 0.875

Conditional variance

1.12. A population consists of smokers and non-smokers. 80% of the population is non-smokers.

Survival time is normally distributed. For smokers, mean survival time is 40 with variance 800. For non-smokers, mean survival time is 45 with variance 600.

Calculate the variance of survival time for an individual randomly selected from the population.

1.13. **[C3 Sample:10]** An insurance company is negotiating to settle a liability claim. If a settlement is not reached, the claim will be decided in the courts 3 years from now.

You are given:

- There is a 50% probability that the courts will require the insurance company to make a payment. The amount of the payment, if there is one, has a lognormal distribution with mean 10 and standard deviation 20.
- In either case, if the claim is not settled now, the insurance company will have to pay 5 in legal expenses, which will be paid when the claim is decided, 3 years from now.
- The most that the insurance company is willing to pay to settle the claim is the expected present value of the claim and legal expenses plus 0.02 times the variance of the present value.
- Present values are calculated using $i = 0.04$.

Calculate the insurance company's maximum settlement value for this claim.

(A) 8.89 (B) 9.93 (C) 12.45 (D) 12.89 (E) 13.53

Additional old CAS Exam 3/3L questions: S06:25,30

Solutions

1.1. The probability of success for n experiments is $1 - 0.98^n$, so profit, ignoring fixed expenses, is

$$1{,}000{,}000(1 - 0.98^n) - 2900n$$

Differentiating this and setting it equal to 0:

$$\begin{aligned}
-10^6(0.98^n)(\ln 0.98) - 2900 &= 0\\
0.98^n &= \frac{-2900}{10^6 \ln 0.98}\\
n &= \frac{\ln \frac{-2900}{10^6 \ln 0.98}}{\ln 0.98} = 96.0815
\end{aligned}$$

Thus either 96 or 97 experiments are needed. Plugging those numbers into the original expression

$$g(n) = 1{,}000{,}000(1 - 0.98^n) - 2900n$$

gets $g(96) = 577{,}818.4$ and $g(97) = 577{,}794.0$, so 96 is best, and the expected profit is $577{,}818.4 - 500{,}000 =$ **77,818.4**. **(A)**

An alternative to calculus which is more appropriate for this discrete exercise is to note that as n increases, at first expected profit goes up and then it goes down. Let X_n be the expected profit with n experiments. Then

$$X_n = 10^6(1 - 0.98^n) - 2900n - 500{,}000$$

and the incremental profit generated by experiment #n is

$$X_n - X_{n-1} = 10^6\left(0.98^{n-1} - 0.98^n\right) - 2900.$$

We want this difference to be greater than 0, which occurs when

$$\begin{aligned}
10^6\left(0.98^{n-1} - 0.98^n\right) &> 2900\\
0.98^{n-1}(0.02) &> 0.0029\\
0.98^{n-1} &> \frac{0.0029}{0.02} = 0.145\\
(n-1)\ln 0.98 &> \ln 0.145\\
n - 1 &< \frac{\ln 0.145}{\ln 0.98} = \frac{-1.93102}{-0.02020} = 95.582
\end{aligned}$$

On the last line, the inequality got reversed because we divided by $\ln 0.98$, a negative number. We conclude that the n^{th} experiment increases profit only when $n < 96.582$, or $n \le 96$, the same conclusion as above.

1.2. The first and second are true by formula (1.4). The third should have squares on the second a and second b, since

$$\operatorname{Var}(aX) = \mathbf{E}[(aX)^2] - \mathbf{E}[aX]^2 = a^2\,\mathbf{E}[X^2] - a^2\,\mathbf{E}[X]^2$$

for example. **(B)**

1.3. The mean of the uniform distribution is $\frac{1}{2}$ and the second moment is $\frac{1}{3}$. So

$$\begin{aligned}\operatorname{Var}(XY) &= \mathbf{E}[X^2Y^2] - \mathbf{E}[X]^2\,\mathbf{E}[Y]^2\\ &= \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\\ &= \frac{1}{9} - \frac{1}{16} = \boxed{\mathbf{\frac{7}{144}}} \qquad \textbf{(D)}\end{aligned}$$

1.4. The number of fatal accidents for each life, N, has a Bernoulli distribution with mean q and variance $q(1-q)$, so the variance in one life's aggregate claims is the variance of bN. $\operatorname{Var}(bN) = b^2\operatorname{Var}(N) = b^2q(1-q) = 2500$. For 100 independent lives, aggregate claims are $100bN$, with variance $100\operatorname{Var}(bN) = 100(2500)$.

For three lives always traveling together, aggregate claims are $3bN$ with variance $3^2\operatorname{Var}(bN) = 9(2500)$. If we add this to the variance of aggregate claims for the other 97 independent lives, the variance is $9(2500) + 97(2500) = 106(2500)$. The expected value of aggregate claims, however, is no different from the expected value of the totally independent group's aggregate claims.

The difference in premiums is therefore

$$\sqrt{106(2500)} - \sqrt{100(2500)} = \boxed{\mathbf{14.7815}} \qquad \textbf{(B)}$$

1.5. From formula (1.4),

$$\begin{aligned}22 = \operatorname{Var}(2X - Y) &= 4(9) + 4 - 2(2)\operatorname{Cov}(X, Y)\\ \operatorname{Cov}(X, Y) &= 4.5\\ \rho_{XY} &= \frac{4.5}{\sqrt{9}\sqrt{4}} = \boxed{\mathbf{0.75}} \qquad \textbf{(D)}\end{aligned}$$

1.6.

$$\begin{aligned}\mathbf{E}[XY] &= (10)(1.20) = 12\\ \mathbf{E}[(XY)^2] &= \big(\mathbf{E}[X^2]\big)\big(\mathbf{E}[Y^2]\big) = \big(10^2 + 100\big)\big(1.20^2 + 0.01\big) = 290\\ \operatorname{Var}(XY) &= 290 - 12^2 = \boxed{\mathbf{146}} \qquad \textbf{(C)}\end{aligned}$$

1.7. For an exponential variable, the variance is the square of the mean. Let θ be the parameter for Y

$$\begin{aligned}\operatorname{Var}(XY) &= \mathbf{E}[X^2]\,\mathbf{E}[Y^2] - \mathbf{E}[X]^2\,\mathbf{E}[Y]^2\\ 7500 &= (25 + 25)(2\theta^2) - 25\theta^2\\ &= 75\theta^2\\ \theta &= 10\\ \operatorname{Var}(Y) &= \theta^2 = \boxed{\mathbf{100}} \qquad \textbf{(C)}\end{aligned}$$

1.8. The mean number of policies sold in one year is $252(0.2) = 50.4$. The variance of the Bernoulli number sold per day is $(0.2)(0.8) = 0.16$, so the variance of the number of policies sold in one year is $252(0.16) = 40.32$. The 95^{th} percentile of the number of policies sold is $50.4 + 1.645\sqrt{40.32} = \boxed{\mathbf{60.85}}$. Rounding this to 61, since an integral number of policies is sold, is appropriate.

1.9. The mean of the uniform is $(70-50)/2 = 10$, and the variance is $120^2/12 = 1200$. Multiply these moments by 50 for 50 policies. The probability that profit is greater than 0, using the normal approximation, is

$$1-\Phi\left(\frac{0-500}{\sqrt{60{,}000}}\right) = \Phi(2.04) = \boxed{\mathbf{0.9793}}$$

1.10. The variance of Die A is

$$\frac{1}{6}\left(\left(1-\frac{7}{2}\right)^2+\left(2-\frac{7}{2}\right)^2+\left(3-\frac{7}{2}\right)^2+\left(4-\frac{7}{2}\right)^2+\left(5-\frac{7}{2}\right)^2+\left(6-\frac{7}{2}\right)^2\right) = \frac{35}{12}.$$

Die B is Bernoulli, only two possibilities with probabilities 1/2 and values 1 and 6, so the variance of one toss is $5^2(1/2)^2 = 25/4$. The variance of the mean is the variance of one toss over n (equation (1.6)). So

$$\begin{aligned}\frac{25}{4n} &< \frac{35}{12}\\ 140n &> 300\\ n &> 2\end{aligned}$$

The answer is $\boxed{3}$. **(C)**

1.11. Use the Law of Total Probability. Let X be the length of time. It's a little easier to calculate the probability that $X > 1/2$.

$$\begin{aligned}\Pr(X>1/2 \mid Y) &= e^{-y/2}\\ \Pr(X>1/2) &= \int_0^\infty 0.25ye^{-y/2}e^{-y/2}\,\mathrm{d}y\\ &= 0.25\int_0^\infty ye^{-y}\,\mathrm{d}y\\ &= 0.25\left(-ye^{-y}\Big|_0^\infty + \int_0^\infty e^{-y}\mathrm{d}y\right)\\ &= 0.25\left(-e^{-y}\Big|_0^\infty\right)\\ &= 0.25\end{aligned}$$

Then $\Pr(X<1/2) = 1-\Pr(X>1/2) = \boxed{\mathbf{0.75}}$. **(D)** (since X is continuous, making $\Pr(X=1/2)=0$).

1.12. Let I be the indicator variable for whether the individual is a smoker. If survival time is T, then

$$\mathrm{Var}(T) = \mathrm{Var}(\mathbf{E}[T\mid I]) + \mathbf{E}[\mathrm{Var}(T\mid I)]$$

The expected value of $T\mid I$ is 40 with probability 0.2 and 45 with probability 0.8. Since it has only two values, it is a Bernoulli variable, and its variance is $\mathrm{Var}(\mathbf{E}[T\mid I]) = (0.2)(0.8)(45-40)^2 = 4$.

The variance of $T\mid I$ is 800 with probability 0.2 and 600 with probability 0.8. The mean of these two values is $\mathbf{E}[\mathrm{Var}(T\mid I)] = 0.2(800)+0.8(600) = 640$.

Thus $\mathrm{Var}(T) = 4+640 = \boxed{\mathbf{644}}$.

As a check, you may calculate the second moment and subtract the first moment squared.

$$\begin{aligned}\mathbf{E}[T] &= \mathbf{E}\big[\mathbf{E}[T\mid I]\big] = 0.2(40)+0.8(45) = 44\\ \mathbf{E}[T^2] &= \mathbf{E}\big[E[T^2\mid I]\big] = 0.2(40^2+800)+0.8(45^2+600) = 2580\\ \mathrm{Var}(T) &= 2580-44^2 = \boxed{\mathbf{644}}\end{aligned}$$

1.13. The expected value of the present value of the claim is $0.5(10/1.04^3)$, and the present value of legal fees is $5/1.04^3$, for a total of $10/1.04^3 = 8.89$. We will compute the variance using the conditional variance formula. The legal expenses are not random and have no variance, so we'll ignore them. Let I be the indicator variable for whether a payment is required, and X the settlement value.

$$\mathrm{Var}(X) = \mathrm{Var}\big(\mathbf{E}[X \mid I]\big) + \mathbf{E}\big[\mathrm{Var}(X \mid I)\big]$$

The expected value of the claim is 0 with probability 50% and $10/1.04^3$ with probability 50%. Thus the expected value can only have one of two values. It is a Bernoulli random variable. The Bernoulli shortcut says that its variance is

$$\mathrm{Var}\big(\mathbf{E}[X \mid I]\big) = (0.5)(0.5)\left(\frac{10}{1.04^3}\right)^2 = 19.7579$$

The variance of the claim is 0 with probability 50% and $(20/1.04^3)^2$ with probability 50%. The expected value of the variance is therefore

$$\mathbf{E}\big[\mathrm{Var}(X \mid I)\big] = (0.5)\left(0 + \left(\frac{20}{1.04^3}\right)^2\right) = 158.0629$$

Therefore, $\mathrm{Var}(X) = 19.7579 + 158.0629 = 177.8208$. The answer is

$$8.89 + 0.02(177.8208) = \boxed{12.4463} \qquad \textbf{(C)}$$

Quiz Solutions

1-1.

$$\mathrm{Var}(X) = (0.7)(0.3)\left(100^2\right) = \boxed{2100}$$

1-2.

$$\mathrm{Var}(X) = \mathbf{E}[\mathrm{Var}(X \mid Y)] + \mathrm{Var}(\mathbf{E}[X \mid Y]) = \mathbf{E}[Y^2] + \mathrm{Var}(Y)$$

Since Y is uniform on $[-10, 2]$, its variance is the range squared over 12, or

$$\mathrm{Var}(Y) = \frac{(-10-2)^2}{12} = 12$$

The second moment of Y is the sum of its variance and the square of its mean. The mean is the midpoint of $[-10, 2]$, or -4. So

$$\mathbf{E}[Y^2] = (-4)^2 + 12 = 28$$
$$\mathrm{Var}(X) = 28 + 12 = \boxed{40}$$

Lesson 2

Survival Distributions: Probability Functions

Reading: *Actuarial Mathematics for Life Contingent Risks* 2nd edition 2.1, 2.2, 2.4, 3.1, 3.11

We will study the probability distribution of future lifetime. Once we have specified the probability distribution, we will be able to answer questions like

> What is the probability that someone age 30 will survive to age 80?
>
> What is the probability that someone age 40 will die between ages 75 and 85?

With regard to the second question, in this course, whenever we say "between ages x and y", we mean between the x^{th} birthday and the y^{th} birthday. To say someone dies between ages 75 and 85 means that the person dies after reaching the 75th birthday, but *before* reaching the 85th birthday. If the person dies one month after his 85th birthday, he has not died between ages 75 and 85.

For our survival models, we will use two styles of notation: probability (the type you use in probability courses, which writes arguments of functions with parentheses after the function symbol, like $f(x)$) and actuarial (which, as you will see, writes arguments as subscripts). We will use actuarial notation most of the time, but since you are probably already familiar with probability notation, we will start by discussing that, and then we'll define actuarial notation in terms of probability notation.

2.1 Probability notation

We first define T_x as the random variable for time to death for someone age x. Thus, for someone age 50, T_{50} is the amount of time until he dies, and to say $T_{50} = 32.4$ means that the person who was originally age 50 died when he was age 82.4, so that he lived exactly 32.4 years. We will use the symbol (x) to mean "someone age x", so (50) means "someone age 50". It is very common in this course to use the letter x to mean age.

In probability notation, $F_T(t)$ is the cumulative distribution function of T, or $\Pr(T \leq t)$. Usually, the cumulative distribution function is called the distribution function, dropping the word "cumulative". As an example of a cumulative distribution function, $F_{T_{50}}(30)$ is the probability that (50) does not survive 30 years. Rather than using a double subscript, we will abbreviate the notation for the cumulative distribution function of T_x as $F_x(t)$, or $F_{50}(30)$ in our example. The complement of the distribution function is called the *survival function* and is denoted by $S_T(t)$. In other words, $S_T(t) = \Pr(T > t)$. Thus $S_{T_{50}}(30)$ is the probability that (50) lives at least 30 years. In general, $S_T(t) = 1 - F_T(t)$. Once again, we'll abbreviate the notation as $S_x(t)$, or $S_{50}(30)$ in our example.

If we wanted to express the probability that (40) will die between ages 75 and 85 in terms of distribution functions, we would write it as

$$\Pr(35 < T_{40} \leq 45) = F_{40}(45) - F_{40}(35)$$

and if we wanted to express it in terms of survival functions, we'd write

$$\Pr(35 < T_{40} \leq 45) = S_{40}(35) - S_{40}(45)$$

In the probability expressions, you may wonder why the inequality is strict on one side and not on the other. The inequalities are strict or non-strict to be consistent with the definitions of F and S. However, we will always assume that T_x is a continuous random variable, so it doesn't matter whether the inequalities are strict or not.

Since we mentioned continuity as a property of our survival functions, let's discuss required and desirable characteristics of a survival function. A survival function must have the following properties:

1. $S_x(0) = 1$. Negative survival times are impossible.
2. $S_x(t) \geq S_x(u)$ for $u > t$. The function is monotonically nonincreasing. The probability of surviving a longer amount of time is never greater than the probability of surviving a shorter amount of time.
3. $\lim_{t\to\infty} S_x(t) = 0$. Eventually everyone dies; T_x is never infinite.

Those are the required properties of a survival function. Examples of valid survival functions (although they may not represent human mortality) are

1. $S_x(t) = e^{-0.01t}$
2. $S_x(t) = \dfrac{x+1}{x+1+t}$
3. $S_x(t) = \begin{cases} 1 - 0.01t & t \leq 100 \\ 0 & t > 100 \end{cases}$

Examples of invalid survival functions are

1. $S_x(t) = \begin{cases} 50 - t & t \leq 50 \\ 0 & t > 50 \end{cases}$. Violates first property.
2. $S_x(t) = |\cos t|$. Violates second and third properties.

We will also assume the following properties for almost all of our survival functions:

1. $S_x(t)$ is differentiable for $t \geq 0$, with at most only a finite number of exceptions. Differentiability will allow us to define the probability density function (except at a finite number of points).
2. $\lim_{t\to\infty} tS_x(t) = 0$. This will assure that mean survival time exists.
3. $\lim_{t\to\infty} t^2 S_x(t) = 0$. This will assure that the variance of survival time exists.

In the three examples of valid survival functions given above, you may verify that the first and third ones satisfy all of these properties but the second one does not satisfy the second and third properties.

Instead of saying "a person age x", we will often use the shorthand "(x)".

We would now like to relate the various T_x variables (one variable for each x) to each other. To do this, note that each T_x is a conditional random variable: it is the distribution of survival time, given that someone survived to age x. We can relate them using conditional probability:

$$\begin{aligned}&\text{Probability } (x) \text{ survives } t+u \text{ years}\\ &\qquad = \text{Probability } (x) \text{ survives } t \text{ years} \times \text{Probability } (x+t) \text{ survives } u \text{ years}\end{aligned}$$

or

$$\begin{aligned}\Pr(T_x > t+u) &= \Pr(T_x > t)\Pr(T_{x+t} > u)\\ S_x(t+u) &= S_x(t)S_{x+t}(u)\\ S_{x+t}(u) &= \frac{S_x(t+u)}{S_x(t)} \qquad (2.1)\end{aligned}$$

In English: If you're given the survival function for (x), and you want to know the probability that someone t years older than x lives at least another u years, calculate the probability of (x) living at least $t+u$ years, and divide by the probability that (x) lives at least t years.

A special case is $x = 0$, for which (changing variables: the new x is the old t, the new t is the old u)

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} \tag{2.2}$$

The corresponding relationship for distribution functions is

$$\Pr(T_x \le t) = \frac{\Pr(T_0 \le x+t) - \Pr(T_0 \le x)}{\Pr(T_0 > x)}$$

$$F_x(t) = \frac{F_0(x+t) - F_0(x)}{1 - F_0(x)} \tag{2.3}$$

Example 2A The survival function for newborns is

$$S_0(t) = \begin{cases} \sqrt{\frac{100-t}{100}} & t \le 100 \\ 0 & t > 100 \end{cases}$$

Calculate

1. The probability that a newborn survives to age 75 but does not survive to age 84.
2. The probability that (20) survives to age 75 but not to age 84.
3. $F_{60}(20)$.

Answer: Write each of the items we want to calculate in terms of survival functions.

1. We want $S_0(75) - S_0(84)$.

$$S_0(75) = \sqrt{\frac{25}{100}} = 0.5$$

$$S_0(84) = \sqrt{\frac{16}{100}} = 0.4$$

$$\Pr(75 < T_0 \le 84) = 0.5 - 0.4 = \boxed{\mathbf{0.1}}$$

2. We want $S_{20}(55) - S_{20}(64)$. We'll use equation (2.1) to calculate the needed survival functions.

$$S_{20}(55) = \frac{S_0(75)}{S_0(20)} = \frac{0.5}{\sqrt{80/100}} = 0.559017$$

$$S_{20}(64) = \frac{S_0(84)}{S_0(20)} = \frac{0.4}{\sqrt{80/100}} = 0.447214$$

$$\Pr(55 < T_{20} \le 64) = 0.559017 - 0.447214 = \boxed{\mathbf{0.111803}}$$

3. $F_{60}(20) = 1 - S_{60}(20)$, and

$$S_{60}(20) = \frac{S_0(80)}{S_0(60)} = \frac{\sqrt{0.2}}{\sqrt{0.4}} = \sqrt{0.5}$$

$$F_{60}(20) = 1 - \sqrt{0.5} = \boxed{\mathbf{0.292893}}$$

□

Quiz 2-1 (40) is subject to the survival function

$$S_{40}(t) = \begin{cases} 1 - 0.005t & t < 20 \\ 1.3 - 0.02t & 20 \le t \le 65 \end{cases}$$

Calculate the probability that (50) survives at least 30 years.

2.2 Actuarial notation

Actuarial notation puts arguments of functions in subscripts and sometimes superscripts before and after the base function symbol instead of using parenthesized arguments. We are going to learn two actuarial functions: p and q right now. Later on in this lesson, we'll also learn two other functions, d and l.

The first function we work with is $S_x(t) = \Pr(T_x > t)$. The actuarial symbol for this is ${}_tp_x$. The letter p denotes the concept of probability of survival. The x subscript is the age; the t presubscript is the duration.

The complement of the survival function is $F_x(t) = \Pr(T_x \le t)$. The actuarial symbol for this is ${}_tq_x$. The letter q denotes the concept of probability of death. A further refinement to this symbol is the probability of delayed death: the probability that T_x is between u and $u + t$, which is denoted by ${}_{u|t}q_x$.[1] For all three symbols, the t (but not the u) is usually omitted if it is 1.

To summarize the notation:

$$\begin{aligned} {}_tp_x &= S_x(t) \\ {}_tq_x &= F_x(t) \\ {}_{u|t}q_x &= F_x(t+u) - F_x(u) = S_x(u) - S_x(t+u) \end{aligned}$$

The following relationships are clear. For each one, an English translation is provided on the right. In these English translations, and indeed throughout this manual, the phrase "survives n years" means that the person does not die within n years. It does *not* mean that the person dies immediately at the end of n years.

$p_x = 1 - q_x$	The probability that a person age x survives one year is 1 minus the probability that the same person dies within one year.
${}_tp_x = 1 - {}_tq_x$	The probability that a person age x survives t years is 1 minus the probability that the same person dies within t years.
${}_tp_x \; {}_up_{x+t} = {}_{t+u}p_x$	The probability that a person age x survives t years and then survives for another u years is the probability that the same person survives $t + u$ years.
${}_{t\|u}q_x = {}_tp_x \; {}_uq_{x+t}$	The probability that a person age x dies at least t years from now but sooner than $t + u$ years from now is the probability that the same person survives t years and then dies within the next u years.

There are two additional useful formulas for ${}_{t|u}q_x$. The probability that (x) dies in the period from t to $t + u$ is the probability that (x) survives t years and does not survive $t + u$ years, or

$$ {}_{t|u}q_x = {}_tp_x - {}_{t+u}p_x \tag{2.4}$$

[1]*Actuarial Mathematics for Life Contingent Risks* uses a large line on the baseline for this symbol, like ${}_{u}|{}_{t}q_x$. I think this is ugly, and older textbooks do not write it this way. Nor do recent exams.

The probability that (x) dies in the period from t to $t+u$ is the probability that (x) dies within $t+u$ years minus the probability that (x) dies within t years, or

$$ {}_{t|u}q_x = {}_{t+u}q_x - {}_tq_x \tag{2.5} $$

Example 2B You are given the following mortality table:

x	q_x
60	0.001
61	0.002
62	0.003
63	0.004
64	0.005

Calculate the probability that a person age 60 will die sometime between 2 and 5 years from now.

Answer: The actuarial notation for what we are calculating is ${}_{2|3}q_{60}$. One way to calculate this is as the probability of living 2 years minus the probability of living 5 years, or ${}_2p_{60} - {}_5p_{60}$. We calculate:

$$\begin{aligned}
{}_2p_{60} &= p_{60}\,p_{61} \\
&= (1-q_{60})(1-q_{61}) = (1-0.001)(1-0.002) = 0.997002 \\
{}_5p_{60} &= {}_2p_{60}\,p_{62}\,p_{63}\,p_{64} \\
&= 0.997002(1-q_{62})(1-q_{63})(1-q_{64}) \\
&= 0.997002(1-0.003)(1-0.004)(1-0.005) = 0.985085 \\
{}_{2|3}q_{60} &= 0.997002 - 0.985085 = \boxed{0.011917}
\end{aligned}$$

□

In the following example, we'll relate actuarial and probability notation.

Example 2C You are given that

$$S_0(t) = \left(\frac{100}{100+t}\right)^2$$

Calculate ${}_{5|}q_{40}$

Answer: First express the desired probability in terms of survival functions, using equation (2.4).

$$ {}_{5|}q_{40} = {}_5p_{40} - {}_6p_{40} = S_{40}(5) - S_{40}(6) $$

Then express these in terms of S_0, using equation (2.2).

$$\begin{aligned}
S_{40}(5) &= \frac{S_0(45)}{S_0(40)} = \frac{(100/145)^2}{(100/140)^2} = 0.932224 \\
S_{40}(6) &= \frac{S_0(46)}{S_0(40)} = \frac{(100/146)^2}{(100/140)^2} = 0.919497
\end{aligned}$$

The answer[2] is ${}_{5|}q_{40} = 0.932224 - 0.919497 = \boxed{0.012727}$ □

[2]Without intermediate rounding, the answer would be 0.012726. We will often show rounded values but use unrounded values in our calculations.

2.3 Life tables

A life table is a concrete way to look at the survivorship random variable. A life table specifies a certain number of lives at a starting integer age x_0. Usually $x_0 = 0$. This number of lives at age x_0 is called the *radix*. Then for each integer $x > x_0$, the expected number of survivors is listed. The notation for the number of lives listed in the table for age x is l_x.

Assuming for simplicity that $x_0 = 0$, the random variable for the number of lives at each age, $\mathcal{L}(x)$, is a binomial random variable with parameters l_0 and ${}_xp_0$, so the expected number of lives is $l_x = l_0\,{}_xp_0$. Similarly, $l_{x+t} = l_x\,{}_tp_x$. More importantly, ${}_tp_x$ can be calculated from the table using ${}_tp_x = l_{x+t}/l_x$.

A life table also lists the expected number of deaths at each age; d_x is the notation for this concept. Thus $d_x = l_x - l_{x+1} = l_x\,q_x$. Therefore, $q_x = d_x/l_x$. The life table also makes it easy to compute ${}_{u|t}q_x = (l_{x+u} - l_{x+t+u})/l_x$.

Here's an example of a life table:

x	l_x	d_x
70	100	10
71	90	15
72	75	15
73	60	20
74	40	18

Notice that on each line, $d_x = l_x - l_{x+1}$. We can deduce that $l_{75} = l_{74} - d_{74} = 40 - 18 = 22$.

EXAMPLE 2D Using the life table above, calculate ${}_3p_{71}$.

ANSWER:

$$ {}_3p_{71} = \frac{l_{74}}{l_{71}} = \frac{40}{90} = \boxed{\frac{4}{9}} $$

□

The notation ${}_nd_x$ is the number of deaths occurring within n years after age x; $d_x = {}_1d_x$, and ${}_nd_x = \sum_{j=0}^{n-1} d_{x+j}$.

On exams, they base questions on the Illustrative Life Table, an abridged version of a table in the Bowers textbook from the previous syllabus. In this table, the column for d_x is omitted, and must be deduced from l_x if needed. However, a column of mortality rates, $1000q_x$, is given, even though mortality rates could also be computed from the l_x's.

Since life tables are so convenient, it is sometimes easier to build a life table to solve a probability question than to work the question out directly.

Continuation of Example 2B. Redo Example 2B using life tables.

ANSWER: We will arbitrarily use a radix of 1,000,000 at age 60. Then we recursively calculate l_x, $x = 61, \ldots, 65$ using $l_{x+1} = l_x(1 - q_x)$.

x	q_x	l_x
60	0.001	1,000,000
61	0.002	999,000
62	0.003	997,002
63	0.004	994,011
64	0.005	990,035
65		985,085

The answer is $(997{,}002 - 985{,}085)/1{,}000{,}000 = \boxed{\mathbf{0.011917}}$. □

An exam question may ask you to fill in the blanks in a life table using probabilities. However, this type of question is probably too simple for SOA exams.

Example 2E You are given the following life table:

x	l_x	d_x	p_x
0		50	
1			0.98
2	890		

Calculate ${}_2p_0$.

Answer: We back out l_0:

$$l_1 = \frac{890}{0.98} = 908.16$$
$$l_0 = 908.16 + 50 = 958.16$$

Hence ${}_2p_0 = 890/958.2 = \boxed{\mathbf{0.9288}}$. □

Quiz 2-2 You are given:

(i) $d_{48} = 80$
(ii) $l_{50} = 450$
(iii) ${}_{3|2}q_{45} = 1/6$
(iv) ${}_3p_{45} = 2/3$

Determine d_{49}.

2.4 Mortality trends

Mortality has been improving in recent years. One may want to assume mortality continues to improve. Thus one may assume the base mortality table applies to a baseline year Y, and set the mortality rate for (x) in year $Y+s$ to $q(x,s)$. If we assume a mortality improvement of a fixed percentage per year, then we can express $q(x,s)$ as

$$q(x,s) = q(x,0)r_x^s$$

where r_x^s, the complement of the mortality improvement rate, is called a *Reduction Factor* in the textbook. In the U.S. this would be called a projection factor, certainly a better name since the factor does not necessarily reduce mortality.

When calculating ${}_tp(x,0)$ in this situation, one must be careful to multiply by the appropriate reduction factors for the *attained ages*; since the attained age varies by year, the reduction factor may vary by year.

Example 2F An excerpt of a mortality table, assumed to apply to 2010 mortality, is:

x	q_x
68	0.01050
69	0.01175
70	0.01340
71	0.01522
72	0.01800

Reduction factors are

$$r_x = \begin{cases} 0.98 & x < 70 \\ 0.99 & x \geq 70 \end{cases}$$

Calculate the probability that someone age 68 on June 30, 2010 will still be alive on June 30, 2015.

ANSWER: The survival probabilities are

$$\begin{aligned} p(68,0) &= 0.98950 \\ p(69,1) &= 1-(0.01175)(0.98) = 0.98849 \\ p(70,2) &= 1-(0.01340)(0.99^2) = 0.98687 \\ p(71,3) &= 1-(0.01522)(0.99^3) = 0.98523 \\ p(72,4) &= 1-(0.01800)(0.99^4) = 0.98271 \end{aligned}$$

Therefore, ${}_5p(68,0) = (0.98950)(0.98849)(0.98687)(0.98523)(0.98271) = \boxed{\mathbf{0.93456}}$. □

Reduction factors have never appeared on an SOA exam, nor do they appear in any sample questions.

Note

Although we've spoken about human mortality throughout this lesson, everything applies equally well to any situation in which you want to study the time of failure random variable. Failure doesn't even have to be a bad thing. Thus, we can study random variables such as time until becoming an FSA, time until first marriage, etc. Define a random variable measuring time until the event of interest, and then you can define the actuarial functions and build a life table.

Exercises

Probability Notation

2.1. **[CAS4-S87:16]** (1 point) You are given the following survival function:

$$S_0(x) = \begin{cases} (10000 - x^2)/10000 & 0 \leq x \leq 100 \\ 0 & x > 100 \end{cases}$$

Calculate q_{32}.

(A) Less than 0.005
(B) At least 0.005, but less than 0.006
(C) At least 0.006, but less than 0.007
(D) At least 0.007, but less than 0.008
(E) At least 0.008

2.2. You are given

(i) $S_{10}(25) = 0.9$
(ii) $F_{20}(15) = 0.05$.

Determine $S_{10}(10)$.

Exercises continue on the next page . . .

Table 2.1: Summary for this lesson

Probability notation

$$F_x(t) = \Pr(T_x \le t)$$
$$S_x(t) = 1 - F_x(t) = \Pr(T_x > t)$$
$$S_{x+u}(t) = \frac{S_x(t+u)}{S_x(u)} \tag{2.1}$$
$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} \tag{2.2}$$
$$F_x(t) = \frac{F_0(x+t) - F_0(x)}{1 - F_0(x)} \tag{2.3}$$

Actuarial notation

${}_tp_x = S_x(t)$ = probability that (x) survives t years
${}_tq_x = F_x(t)$ = probability that (x) dies within t years
${}_{t|u}q_x$ = probability that (x) survives t years and then dies in the next u years.

$${}_{t+u}p_x = {}_tp_x\ {}_up_{x+t}$$
$${}_{t|u}q_x = {}_tp_x\ {}_uq_{x+t}$$
$$= {}_tp_x - {}_{t+u}p_x \tag{2.4}$$
$$= {}_{t+u}q_x - {}_tq_x \tag{2.5}$$

Life table functions

l_x is the number of lives at exact age x.
d_x is the number of deaths at age x; in other words, the number of deaths between exact age x and exact age $x+1$.
${}_nd_x$ is the number of deaths between exact age x and exact age $x+n$.

$${}_tp_x = \frac{l_{x+t}}{l_x}$$
$${}_tq_x = \frac{{}_td_x}{l_x} = \frac{l_x - l_{x+t}}{l_x}$$
$${}_{t|u}q_x = \frac{{}_ud_{x+t}}{l_x} = \frac{l_{x+t} - l_{x+t+u}}{l_x}$$

Mortality trends

$$q(x,s) = q(x,0)\,r_x^s$$

Exercises continue on the next page . . .

2.3. You are given the survival function

$$S_0(t) = \frac{t^2 - 190t + 9000}{9000} \quad t \leq 90$$

Determine the probability that a life currently age 36 dies between ages 72 and 81.

Actuarial Notation

2.4. **[CAS4-S88:16]** (1 point) Which of the following are equivalent to ${}_tp_x$?

(A) ${}_{t|u}q_x - {}_{t+u}p_x$
(B) ${}_{t+u}q_x - {}_tq_x + {}_{t+u}p_x$
(C) ${}_tq_x - {}_{t+u}q_x + {}_tp_{x+u}$
(D) ${}_tq_x - {}_{t+u}q_x - {}_tp_{x+u}$
(E) The correct answer is not given by (A), (B), (C), or (D).

2.5. You are given:

(i) The probability that a person age 50 is alive at age 55 is 0.9.
(ii) The probability that a person age 55 is not alive at age 60 is 0.15.
(iii) The probability that a person age 50 is alive at age 65 is 0.54.

Calculate the probability that a person age 55 dies between ages 60 and 65.

2.6. **[4-S86:13]** You are given that ${}_{t|}q_x = 0.10$ for $t = 0, 1, \ldots, 9$.
Calculate ${}_2p_{x+5}$.

(A) 0.40 (B) 0.60 (C) 0.72 (D) 0.80 (E) 0.81

2.7. **[150-82-94:10]** You are given the following:

(i) The probability that a person age 20 will survive 30 years is 0.7.
(ii) The probability that a person age 45 will die within 5 years and that another person age 40 will survive 5 years is 0.0475.
(iii) The probability that a person age 20 will survive 20 years and that another person age 40 will die within 5 years is 0.04.

Calculate the probability that a person age 20 will survive 25 years.

(A) 0.74 (B) 0.75 (C) 0.76 (D) 0.77 (E) 0.78

Exercises continue on the next page . . .

2.8. **[CAS4A-S98:13]** (2 points) You are given the following information:

1. The probability that two 70-year-olds are both alive in 20 years is 16%.
2. The probability that two 80-year-olds are both alive in 20 years is 1%.
3. There is an 8% chance of a 70-year-old living 30 years.
4. All lives are independent and have the same expected mortality.

Determine the probability of an 80-year-old living 10 years.

(A) Less than 0.35
(B) At least 0.35, but less than 0.45
(C) At least 0.45, but less than 0.55
(D) At least 0.55, but less than 0.65
(E) At least 0.65

Life Tables

2.9. You are given the following mortality table:

x	l_x	d_x	${}_{x-60\mid}q_{60}$
60	1000		
61		100	
62			0.07
63	780		

Calculate q_{60}.

2.10. **[CAS4A-S93:2]** (1 point) You are given the following information:

$$\begin{aligned} l_1 &= 9700 \\ q_1 = q_2 &= 0.020 \\ q_4 &= 0.026 \\ d_3 &= 232 \end{aligned}$$

Determine the expected number of survivors to age 5.

(A) Less than 8,845
(B) At least 8,845, but less than 8,850
(C) At least 8,850, but less than 8,855
(D) At least 8,855, but less than 8,860
(E) At least 8,860

Exercises continue on the next page . . .

2.11. **[CAS4A-F93:1]** (1 point) You are given the following mortality table:

x	q_x	l_x	d_x
20		30,000	1,200
21			
22		27,350	
23	0.0700		
24	0.0790	23,900	

Determine the probability that a life age 21 will die within two years.

(A) Less than 0.0960
(B) At least 0.0960, but less than 0.1010
(C) At least 0.1010, but less than 0.1060
(D) At least 0.1060, but less than 0.1110
(E) At least 0.1110

2.12. Jack enters a mortality study at age 25. He dies between ages 65 and 67.

Which of the following does *not* express the likelihood of this event?

(A) ${}_{40}p_{25} \cdot {}_{2}q_{65}$
(B) $\dfrac{S_0(65) - S_0(67)}{S_0(25)}$
(C) ${}_{40}p_{25} - {}_{42}p_{25}$
(D) $\dfrac{d_{66} + d_{67}}{l_{25}}$
(E) ${}_{40|2}q_{25}$

2.13. **[CAS4A-S92:4]** (2 points) You are given the following mortality table:

x	l_x	q_x	d_x
50	1,000	0.020	
51			32
52			30
53			28
54		0.028	

In a group of 800 people age 50, determine the expected number who will die while age 54.

(A) Less than 21
(B) At least 21, but less than 24
(C) At least 24, but less than 27
(D) At least 27, but less than 30
(E) At least 30

Exercises continue on the next page . . .

2.14. **[CAS4A-S99:12]** (2 points) Given the following portion of a life table:

x	l_x	d_x	p_x	q_x
0	1,000	—	0.875	—
1	—	—	—	—
2	750	—	—	0.25
3	—	—	—	—
4	—	—	—	—
5	200	120	—	—
6	—	—	—	—
7	—	20	—	1.00

Determine the value of $p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot q_6$.

(A) Less than 0.055
(B) At least 0.055, but less than 0.065
(C) At least 0.065, but less than 0.075
(D) At least 0.075
(E) The answer cannot be determined from the given information.

2.15. **[3-S00:28]** For a mortality study on college students:

(i) Students entered the study on their birthdays in 1963.
(ii) You have no information about mortality before birthdays in 1963.
(iii) Dick, who turned 20 in 1963, died between his 32$^{\text{nd}}$ and 33$^{\text{rd}}$ birthdays.
(iv) Jane, who turned 21 in 1963, was alive on her birthday in 1998, at which time she left the study.
(v) All lifetimes are independent.
(vi) Likelihoods are based upon the Illustrative Life Table.

Calculate the likelihood for these two students.

(A) 0.00138 (B) 0.00146 (C) 0.00149 (D) 0.00156 (E) 0.00169

2.16. Mortality follows the Illustrative Life Table. Jack and Jill are two independent lives of ages 25 and 30 respectively.

Calculate the probability of Jack and Jill both living to at least age 65 but not to age 90.

Exercises continue on the next page . . .

2.17. **[CAS4A-F98:15]** (2 points) Light bulbs burn out according to the following life table:

l_0	1,000,000
l_1	800,000
l_2	600,000
l_3	300,000
l_4	0

A new plant has 2,500 light bulbs. Burned out light bulbs are replaced with new light bulbs at the end of each year.

What is the expected number of new light bulbs that will be needed at the end of year 3?

(A) Less than 800
(B) At least 800, but less than 860
(C) At least 860, but less than 920
(D) At least 920, but less than 980
(E) At least 980

2.18. You are given the following life table:

x	l_x	d_x
80	5000	59
81	4941	77
82	4864	74
83	4790	80

Let X be the number of survivors to age 83 from a cohort of 5000 lives at age 80.

Calculate $\mathrm{Var}(X)$.

2.19. **[M-F05:31]** The graph of a piecewise linear survival function, $S_0(t)$, consists of 3 line segments with endpoints (0,1), (25,0.50), (75,0.40), (100,0).

Calculate $\dfrac{{}_{20|55}q_{15}}{{}_{55}q_{35}}$.

(A) 0.69 (B) 0.71 (C) 0.73 (D) 0.75 (E) 0.77

2.20. You are given

(i) In year 2012, $l_x = 100 - x$
(ii) Reduction factors are 0.95 for $x < 50$, 0.98 for $x \geq 50$.

Calculate the probability that a life age 48 in 2012 will die between ages 50 and 52.

Additional old SOA Exam MLC questions: F13:24, S14:1, S15:1, F15:1,2,B1(a), F16:2

Additional old CAS Exam 3/3L questions: S05:29, S07:5, S08:14, F08:14, S09:2, F09:1, S10:2, F10:2, S11:2, S12:4, S13:1, F13:1

Additional old CAS Exam LC questions: S14:3, F14:1, S15:1, S16:2,4

Solutions

2.1. Translate q_{32} into survival functions and use using equation (2.2).

$$\begin{aligned} q_{32} &= F_{32}(1) = 1 - S_{32}(1) \\ S_{32}(1) &= \frac{S_0(33)}{S_0(32)} \\ &= \frac{10000 - 33^2}{10000 - 32^2} = 0.992758 \\ F_{32}(1) &= 1 - 0.992758 = \boxed{0.007242} \qquad \textbf{(D)} \end{aligned}$$

2.2. The probability of (10) surviving 25 years is the probability of (10) surviving 10 years and then another 15 years.

$$\begin{aligned} S_{10}(25) &= S_{10}(10) S_{20}(15) \\ 0.9 &= S_{10}(10)(1 - 0.05) \\ S_{10}(10) &= \frac{0.9}{0.95} = \boxed{0.9474} \end{aligned}$$

2.3. We will need $S_0(36)$, $S_0(72)$, and $S_0(81)$.

$$\begin{aligned} S_0(36) &= \frac{36^2 - 190(36) + 9000}{9000} = 0.384 \\ S_0(72) &= \frac{72^2 - 190(72) + 9000}{9000} = 0.056 \\ S_0(81) &= \frac{81^2 - 190(81) + 9000}{9000} = 0.019 \\ {}_{36|9}q_{36} &= {}_{36}p_{36} - {}_{45}p_{36} \\ &= \frac{S_0(72)}{S_0(36)} - \frac{S_0(81)}{S_0(36)} \\ &= \frac{0.056}{0.384} - \frac{0.019}{0.384} = \boxed{0.096354} \end{aligned}$$

2.4. Using ${}_{t|u}q_x = {}_tp_x - {}_{t+u}p_x$, (A) becomes ${}_tp_x - 2\,{}_{t+u}p_x$, so it doesn't work.

In (B), we note that ${}_{t+u}q_x + {}_{t+u}p_x = 1$, so it becomes $1 - {}_tq_x = {}_tp_x$, the correct answer.

(C) and (D) each have ${}_tp_{x+u}$ which is a function of survival to age $x + u$, and no other term to cancel it, so those expressions cannot possibly equal ${}_tp_x$ which only depends on survival to age x.

(B)

2.5. We need ${}_{5|5}q_{55} = {}_5p_{55} - {}_{10}p_{55}$.

$$\begin{aligned} {}_5p_{55} &= 1 - {}_5q_{55} = 1 - 0.15 = 0.85 \\ {}_{15}p_{50} &= 0.54 \\ {}_5p_{50}\,{}_{10}p_{55} &= 0.54 \\ 0.9\,{}_{10}p_{55} &= 0.54 \\ {}_{10}p_{55} &= 0.6 \end{aligned}$$

The answer is $0.85 - 0.6 = \boxed{0.25}$.

2.6. It's probably easiest to go from ${}_{t|}q_x$ to ${}_2p_{x+5}$, which is what we need, by using a life table. If we start with a radix of $l_x = 10$, then since $1/10$ of the population dies each year, $l_{x+t} = 10 - t$. Then

$$ {}_2p_{x+5} = \frac{l_{x+7}}{l_{x+5}} = \frac{10-7}{10-5} = \boxed{0.6} \qquad \textbf{(B)} $$

2.7. Perhaps the best way to do this exercise is to construct a life table. Let $l_{20} = 1$. By (i), $l_{50} = 0.7$. In (iii), we are given that the following is equal to 0.04:

$$ {}_{20}p_{20}\,{}_5q_{40} = \left(\frac{l_{40}}{l_{20}}\right)\left(\frac{l_{40} - l_{45}}{l_{40}}\right) = \frac{l_{40} - l_{45}}{l_{20}} $$

and since we set $l_{20} = 1$, we have $l_{40} - l_{45} = 0.04$. Let $x = l_{45}$, which incidentally is the final answer we're looking for, since we're looking for ${}_{25}p_{20} = l_{45}/l_{20}$ and we've set $l_{20} = 1$. The expression that is equal to 0.0475 by (ii) is

$$ {}_5q_{45}\,{}_5p_{40} = \left(\frac{l_{45} - l_{50}}{l_{45}}\right)\left(\frac{l_{45}}{l_{40}}\right) = \frac{l_{45} - l_{50}}{l_{40}} = \frac{x - 0.7}{x + 0.04} $$

Let's solve for x.

$$ \begin{aligned} x - 0.7 &= 0.0475(x + 0.04) \\ 0.9525x &= 0.7 + 0.04(0.0475) = 0.7019 \\ x &= \boxed{0.736903} \qquad \textbf{(A)} \end{aligned} $$

2.8. There are three variables: $x = {}_{10}p_{70}$, $y = {}_{10}p_{80}$, and $z = {}_{10}p_{90}$. We are given
1. $(xy)^2 = 0.16 \Rightarrow xy = 0.4$
2. $(yz)^2 = 0.01 \Rightarrow yz = 0.1$
3. $xyz = 0.08$

and we want y. From the first and third statement, $z = 0.08/0.4 = 0.2$. Then from the second statement, $y = 0.1/0.2 = \boxed{0.5}$. **(C)**

2.9. Did you notice that you are given ${}_{x-60|}q_{60}$ rather than q_x?

Since ${}_{2|}q_{60} = 0.07$, then $d_{62} = 0.07l_{60} = 70$ and $l_{62} = l_{63} + d_{62} = 780 + 70 = 850$. Then $l_{61} = 850 + d_{61} = 950$ and $d_{60} = l_{60} - l_{61} = 1000 - 950 = 50$, so $q_{60} = d_{60}/l_{60} = 50/1000 = \boxed{0.05}$.

2.10. We recursively compute l_x through $x = 5$.

$$ \begin{aligned} l_2 &= l_1(1 - q_1) = 9700(1 - 0.020) = 9506 \\ l_3 &= l_2(1 - q_2) = 9506(1 - 0.020) = 9315.88 \\ l_4 &= l_3 - d_3 = 9315.88 - 232 = 9083.88 \\ l_5 &= l_4(1 - q_4) = 9083.88(1 - 0.026) = \boxed{8847.70} \qquad \textbf{(B)} \end{aligned} $$

2.11. We need l_{21} and l_{23}.

$$ \begin{aligned} l_{21} &= 30{,}000 - 1{,}200 = 28{,}800 \\ l_{23} &= 23{,}900/(1 - 0.0700) = 25{,}698.92 \\ {}_2q_{21} &= \frac{28{,}800 - 25{,}698.92}{28{,}800} = \boxed{0.1077} \qquad \textbf{(D)} \end{aligned} $$

2.12. All of these expressions are fine except for **(D)**, which should have $d_{65}+d_{66}$ in the numerator.

2.13. $l_{54} = 1000(1-0.020)-32-30-28 = 890$. Then ${}_{4|}q_{50} = (890/1000)(0.028) = 0.02492$. For 800 people, $800(0.02492) = \boxed{19.936}$. **(A)**

2.14. They gave you superfluous information for year 2. It's the usual CAS type of humor—it's not that the answer cannot be determined from the given information (almost never the right answer choice), rather there is too much information provided.

We back out $l_7 = 20$, since everyone dies that year. We calculate $l_6 = l_5 - d_5 = 80$. $l_1 = 1000(0.875) = 875$. Then what the question is asking for is

$$ {}_{5|}q_1 = \frac{l_6 - l_7}{l_1} = \frac{80-20}{875} = \boxed{0.06857} \qquad \textbf{(C)} $$

2.15. For independent lifetimes, we multiply the likelihood for each life together to get the likelihood of the joint event.

For Dick, the condition is age 20, and death occurs at age 32, so we need $\dfrac{d_{32}}{l_{20}} = \dfrac{l_{32}-l_{33}}{l_{20}}$.

For Jane, the condition is age 21 and she survived to age 56, so we need $\dfrac{l_{56}}{l_{21}}$.

Looking up the Illustrative Life Table, we find

x	l_x
20	9,617,802
21	9,607,896
32	9,471,591
33	9,455,522
56	8,563,435

The answer is

$$ \left(\frac{9{,}471{,}591 - 9{,}455{,}522}{9{,}617{,}802}\right)\left(\frac{8{,}563{,}435}{9{,}607{,}896}\right) = \boxed{0.001489} \qquad \textbf{(C)} $$

2.16. For Jack, we need $(l_{65}-l_{90})/l_{25}$, and for Jill we need $(l_{65}-l_{90})/l_{30}$. From the Illustrative Life Table, we have

x	l_x
25	9,565,017
30	9,501,381
65	7,533,964
90	1,058,491

$$ \left(\frac{7{,}533{,}964 - 1{,}058{,}491}{9{,}565{,}017}\right)\left(\frac{7{,}533{,}964 - 1{,}058{,}491}{9{,}501{,}381}\right) = \boxed{0.4614} $$

2.17. We have to keep track of three cohorts of light bulbs, the ones installed at times 0, 1, and 2. From the life table, the unconditional probabilities of failure are $q_0 = 0.2$ in year 0, ${}_{1|}q_0 = 0.2$ in year 1, and ${}_{2|}q_0 = 0.3$ in year 3. Of the 2500 original bulbs, 500 apiece fail in years 1 and 2 and 750 in year 3. 500 new ones are installed in year 1, and 100 apiece fail in years 2 and 3. Finally $500+100 = 600$ are installed in year 2, of which 120 fail in year 3. The answer is $750 + 100 + 120 = 970$. **(D)**. Here's a table with these results:

	Number installed	Failures year 1	Failures year 2	Failures year 3
Installed in year 0	2500	500	500	750
Installed in year 1	500		100	100
Installed in year 2	600			120
Total failures		500	600	970

2.18. X is binomial with parameters 5000 and ${}_3p_{80} = 4790/5000$, so its variance is

$$5000\left(\frac{4790}{5000}\right)\left(1-\frac{4790}{5000}\right) = \boxed{201.18}$$

The variance is $l_{80}\,{}_3p_{80}\,{}_3q_{80}$. Notice that the variance of the number who die in the three years is the same.

2.19. The given fraction is

$$\frac{{}_{20|55}q_{15}}{{}_{55}q_{35}} = \frac{{}_{20}p_{15}\,{}_{55}q_{35}}{{}_{55}q_{35}} = {}_{20}p_{15} = \frac{S_0(35)}{S_0(15)}$$

By linear interpolation, $S_0(15) = 0.7$ and $S_0(35) = 0.48$. So the quotient is $0.48/0.7 = \boxed{0.685714}$. **(A)**

2.20. First, $q(x,0) = (l_x - l_{x+1})/l_x = 1/(100-x)$, so $q(48,0) = 1/52$, $q(49,0) = 1/51$, $q(50,0) = 1/50$, and $q(51,0) = 1/49$. Then

$$\begin{aligned}
p(48,0) &= 1 - 1/52 = 0.980769\\
p(49,1) &= 1 - 0.95/51 = 0.981373\\
p(50,2) &= 1 - 0.98^2/50 = 0.980792\\
p(51,3) &= 1 - 0.98^3/49 = 0.980792\\
{}_2p(48,0) &= (0.980769)(0.981373) = 0.962500\\
{}_2p(50,2) &= (0.980792)(0.980792) = 0.961953\\
{}_{2|2}q(48,0) &= 0.962500(1 - 0.961953) = \boxed{0.03662}
\end{aligned}$$

Quiz Solutions

2-1. Use equation (2.1) to relate S_{50} to S_{40}.

$$\begin{aligned}
S_{50}(30) &= \frac{S_{40}(40)}{S_{40}(10)}\\
S_{40}(40) &= 1.3 - 0.02(40) = 0.5\\
S_{40}(10) &= 1 - 0.005(10) = 0.95\\
S_{50}(30) &= \frac{0.5}{0.95} = \boxed{0.5263}
\end{aligned}$$

2-2. Since ${}_{3|2}q_{45} = {}_3p_{45}\,{}_2q_{48}$, we deduce ${}_2q_{48} = (1/6)/(2/3) = 1/4$, and therefore $l_{50}/l_{48} = 1 - {}_2q_{48} = 3/4$ and $l_{48} = 450(4/3) = 600$. Then $l_{50} = l_{48} - d_{48} - d_{49}$, or $450 = 600 - 80 - d_{49}$, implying $d_{49} = \boxed{70}$.

Lesson 3

Survival Distributions: Force of Mortality

Reading: *Actuarial Mathematics for Life Contingent Risks* 2nd edition 2.3 As discussed in the previous lesson, we are assuming that the distribution function of the time-to-death random variable T_x, or $F_x(t)$, is differentiable, and therefore the probability density function of the time-to-death random variable, or $f_x(t)$, is

$$f_x(t) = \frac{\mathrm{d}F_x(t)}{\mathrm{d}t} = -\frac{\mathrm{d}S_x(t)}{\mathrm{d}t}$$

We can derive F or S from f by integrating:

$$F_x(t) = \int_0^t f_x(u)\mathrm{d}u$$
$$S_x(t) = \int_t^\infty f_x(u)\mathrm{d}u$$

A function related to density is the *force of mortality*, denoted by μ_x. For the time-to-death random variable T_x, this is the hazard rate function discussed in Lesson 1, in the third bullet on page 1. It measures the rate of mortality at age x, given survival to age x. It can also be defined as the probability of death in a small amount of time after age x, given survival to age x, or

$$\mu_x = \lim_{\mathrm{d}x\to 0^+} \frac{1}{\mathrm{d}x} \Pr(T_0 \le x + \mathrm{d}x \mid T_0 > x)$$

Now,

$$\Pr(T_0 \le x + h \mid T_0 > x) = \frac{S_0(x) - S_0(x+h)}{S_0(x)}$$

so the force of mortality can be written as

$$\begin{aligned}\mu_x &= \frac{1}{S_0(x)} \lim_{h\to 0} \frac{S_0(x) - S_0(x+h)}{h}\\ &= -\frac{\mathrm{d}S_0(x)/\mathrm{d}x}{S_0(x)} && (*)\\ &= \frac{f_0(x)}{S_0(x)} && (**)\end{aligned}$$

Since $S_x(t) = S_0(x+t)/S_0(x)$, differentiating this expression with respect to t, we get

$$f_x(t) = f_0(x+t)/S_0(x)$$

It follows that

$$\frac{f_x(t)}{S_x(t)} = \frac{f_0(x+t)/S_0(x)}{S_0(x+t)/S_0(x)} = \frac{f_0(x+t)}{S_0(x+t)}$$

and by (**)

$$\mu_{x+t} = \frac{f_0(x+t)}{S_0(x+t)} = \frac{f_x(t)}{S_x(t)}$$

Therefore

$$\mu_{x+t} = \frac{f_x(t)}{S_x(t)} \tag{3.1}$$

regardless of which x or t you pick. As long as you hold the sum $x+t$ fixed, you're free to pick any x and t you wish in order to calculate μ_{x+t}. For example, if you want to calculate μ_{50} and you are given probability functions $f_{30}(t)$ and $S_{30}(t)$, you may calculate $\mu_{50} = f_{30}(20)/S_{30}(20)$.

In the Bowers textbook formerly on the syllabus, despite the fact that μ_{x+t} is only a function of $x+t$ and not a function of x and t separately, they used the notation $\mu_x(t)$ to mean the same as what we mean by μ_{x+t}. This emphasizes that x is generally held fixed and only t varies. For example, $\mu_{50} = \mu_{30}(20) = \mu_{15}(35)$. *The "default" value of the argument t is 0, not 1.* μ_{20} is not $\mu_{20}(1)$. Sometimes, the notation $\mu(x)$ is used to mean μ_x. In this manual, we will almost always use the notation μ_x and not $\mu_x(t)$ or $\mu(x)$, and the exam will always use that notation.

Since $S_x(t) = {}_tp_x$ and $F_x(t) = {}_tq_x$ so that $f_x(t) = \mathrm{d}\,{}_tq_x/\mathrm{d}t$, formula (3.1) can be written as

$$\mu_{x+t} = \frac{\mathrm{d}\,{}_tq_x/\mathrm{d}t}{{}_tp_x} \tag{3.2}$$

The derivative of the logarithm of a function equals the derivative of the function divided by the function:

$$\frac{\mathrm{d}\ln h(x)}{\mathrm{d}x} = \frac{\mathrm{d}h(x)/\mathrm{d}x}{h(x)} \quad \text{for any function } h$$

Therefore from (*), we have

$$\mu_x = -\frac{\mathrm{d}\ln S_0(x)}{\mathrm{d}x}$$

or

$$\mu_{x+t} = -\frac{\mathrm{d}\ln S_x(t)}{\mathrm{d}t} \tag{3.3}$$

$$= -\frac{\mathrm{d}\ \ln {}_tp_x}{\mathrm{d}t} \tag{3.4}$$

where we're free to choose any x and t that add up to the subscript of μ. Going in the other direction

$$S_x(t) = \exp\left(-\int_0^t \mu_{x+s}\,\mathrm{d}s\right) \tag{3.5}$$

$${}_tp_x = \exp\left(-\int_0^t \mu_{x+s}\,\mathrm{d}s\right) \tag{3.6}$$

$$= \exp\left(-\int_x^{x+t} \mu_s\,\mathrm{d}s\right) \tag{3.7}$$

The only difference between the last two expressions is the variable of integration; otherwise they are equivalent.

What you should remember about equations (3.6) and (3.7) is *if you integrate μ_x from a lower bound to an upper bound, and then exponentiate the negative of the integral, you get the probability of survival to the age represented by the upper bound, conditional on survival to the age represented by the lower bound*. You can calculate ${}_tq_x$ as the complement of an exponentiated integral, but you cannot calculate ${}_{t|u}q_x$ directly in this fashion. Instead, you express it as ${}_tp_x - {}_{t+u}p_x$ and calculate the two survival probabilities in terms of μ_s.

Example 3A You are given[1] that $\mu_{35+t} = 1/(100+t)$.

1. Calculate ${}_{10}p_{35}$.
2. Calculate ${}_{20}q_{45}$.
3. Calculate ${}_{10|20}q_{40}$.

Answer: 1. We want the probability of survival to 45 given survival to 35, so we'll integrate μ from age 35 to age 45. $\mu_{35}(0)$ represents age 35 and $\mu_{35}(10)$ represents age 45.

$$\begin{aligned}{}_{10}p_{35} &= \exp\left(-\int_0^{10} \mu_{35+s}\,ds\right)\\ &= \exp\left(-\int_0^{10} \frac{ds}{100+s}\right)\\ &= \exp\bigl(-(\ln 110 - \ln 100)\bigr)\\ &= \exp \ln \frac{100}{110} = \boxed{\frac{10}{11}}\end{aligned}$$

2. We calculate the probability of survival to 65 given survival to 45 and then take the complement.

$$\begin{aligned}{}_{20}p_{45} &= \exp\left(-\int_{10}^{30} \mu_{35+s}\,ds\right)\\ &= \exp\left(-\int_{10}^{30} \frac{ds}{100+s}\right)\\ &= \exp\bigl(-(\ln 130 - \ln 110)\bigr)\\ &= \exp \ln \frac{110}{130} = \frac{11}{13}\\ {}_{20}q_{45} &= \boxed{\frac{2}{13}}\end{aligned}$$

3. This can be evaluated as ${}_{10}p_{40} - {}_{30}p_{40}$ or as ${}_{10}p_{40}\,{}_{20}q_{50}$; either way, we need two integrals to evaluate this. We'll use the former expression. We already saw in the previous two solutions that for this force of mortality, ${}_tp_x = (65+x)/(65+x+t)$.

$$\begin{aligned}{}_{10|20}q_{40} &= {}_{10}p_{40} - {}_{30}p_{40}\\ &= \frac{65+40}{65+50} - \frac{65+40}{65+70}\\ &= \frac{105}{115} - \frac{105}{135} = \boxed{0.135266}\end{aligned}$$

□

[1]This is a Pareto distribution, a poor model for human mortality, and is used only to illustrate the concept of calculating probabilities from the force of mortality. In fact, the survival function violates the second and third desirable properties for survival functions mentioned on page 20.

Quiz 3-1 The force of mortality is $\mu_x = 0.001\sqrt{x}$.

Calculate the probability of someone age 20 surviving 30 years and then dying in the next 10 years.

By equation (3.1), the density function for T_x is

$$f_x(t) = S_x(t)\mu_{x+t} = {}_tp_x\,\mu_{x+t} \tag{3.8}$$

Thus, probabilities of T_x being in a given range can be computed by integrating ${}_tp_x\,\mu_{x+t}$ over that range:

$$\Pr\bigl(t < T_x \le t+u\bigr) = {}_{t|u}q_x = \int_t^{t+u} {}_sp_x\,\mu_{x+s}\,\mathrm{d}s \tag{3.9}$$

and in particular,

$$ {}_tq_x = \int_0^t {}_sp_x\,\mu_{x+s}\,\mathrm{d}s \tag{3.10}$$

In order for μ_x to be a legitimate force of mortality, the survival function must go to zero as x goes to infinity, which by equation (3.5) means that $\int_0^x \mu_t\,\mathrm{d}t$ must go to infinity as x goes to infinity. μ_x itself does not have to go to infinity, although for a realistic mortality function it would keep increasing as x increases for x greater than 30 or so.

You should understand how a linear transformation of μ affects p. Look at equation (3.6). Since μ is exponentiated to get p, adding something to μ corresponds to multiplying ${}_tp_x$ by e to negative t times that something. Multiplying μ by a constant corresponds to raising p to that power. In other words,

- Suppose one person has force of mortality μ_{x+s} for $0 \le s \le t$ and survival probability ${}_tp_x$ and another one has force of mortality μ'_{x+s} for $0 \le s \le t$ and survival probability ${}_tp'_x$. Then a person who has force of mortality $\mu_{x+s} + \mu'_{x+s}$ for $0 \le s \le t$ has survival probability ${}_tp_x\,{}_tp'_x$. In particular, if μ'_{x+s} is constant k for all s, then the survival probability for the third person is $e^{-kt}\,{}_tp_x$.

- If a force of mortality μ_{x+s}, $0 \le s \le t$, results in survival probability ${}_tp_x$, then a force of mortality of $k\mu_{x+s}$, $0 \le s \le t$ will result in survival probability $({}_tp_x)^k$.

Note that the force of mortality must be multiplied or added throughout an entire range of values $0 \le s \le t$ in order for these results to be true.

Example 3B Sarah's force of mortality is μ_x, and her probability of dying at age 70, q_{70}, is 0.01. Toby's force of mortality is $\mu'_x = 0.5\mu_x + 0.1$.

Calculate Toby's probability of dying at age 70.

Answer: As discussed above, multiplying the force of mortality by 0.5 results in raising the probability of survival to the 0.5 power. Adding 0.1 to the force of mortality for one year multiplies the survival probability by $e^{-0.1}$. So the answer will be $1 - (1-0.01)^{0.5}e^{-0.1}$. Once you get used to this, you will not have to carry out the math. But this time around, let's work it out step by step.

Denoting Toby's functions with primes, we have

$$
\begin{aligned}
p'_{70} &= \exp\left(-\int_0^1 \mu'_{70+t}\,\mathrm{d}t\right) && \text{by equation (3.6) applied to primed functions}\\
&= \exp\left(-\int_0^1 (0.5\mu_{70+t}+0.1)\mathrm{d}t\right) && \text{using the information given in the example to expand } \mu'\\
&= \left(\exp\left(-\int_0^1 0.5\mu_{70+t}\,\mathrm{d}t\right)\right)\left(\exp\left(-\int_0^1 0.1\,\mathrm{d}t\right)\right)\\
&= \left(\exp\left(-0.5\int_0^1 \mu_{70+t}\,\mathrm{d}t\right)\right)\left(e^{-0.1}\right)\\
&= \left(\exp\left(-\int_0^1 \mu_{70+t}\,\mathrm{d}t\right)\right)^{0.5}\left(e^{-0.1}\right)\\
&= \left(p_{70}^{0.5}\right)\left(e^{-0.1}\right) && \text{by equation (3.6) applied to unprimed functions}\\
&= \left((1-q_{70})^{0.5}\right)\left(e^{-0.1}\right)\\
&= (0.99^{0.5})(e^{-0.1})\\
&= 0.90030
\end{aligned}
$$

Hence, $q'_{70} =$ **0.09970**. □

Exam questions will expect you to go between μ and S, p, or q, in either direction. Table 3.1 reviews the relationships between these functions. Figure 3.1 diagrams the relationships, with the actuarial notation on top of each circle and the mathematical notation on the bottom.

Exercises

3.1. A person age 70 is subject to the following force of mortality:

$$\mu_{70+t} = \begin{cases} 0.01 & t \le 5 \\ 0.02 & t > 5 \end{cases}$$

Calculate ${}_{20}p_{70}$ for this person.

Exercises continue on the next page . . .

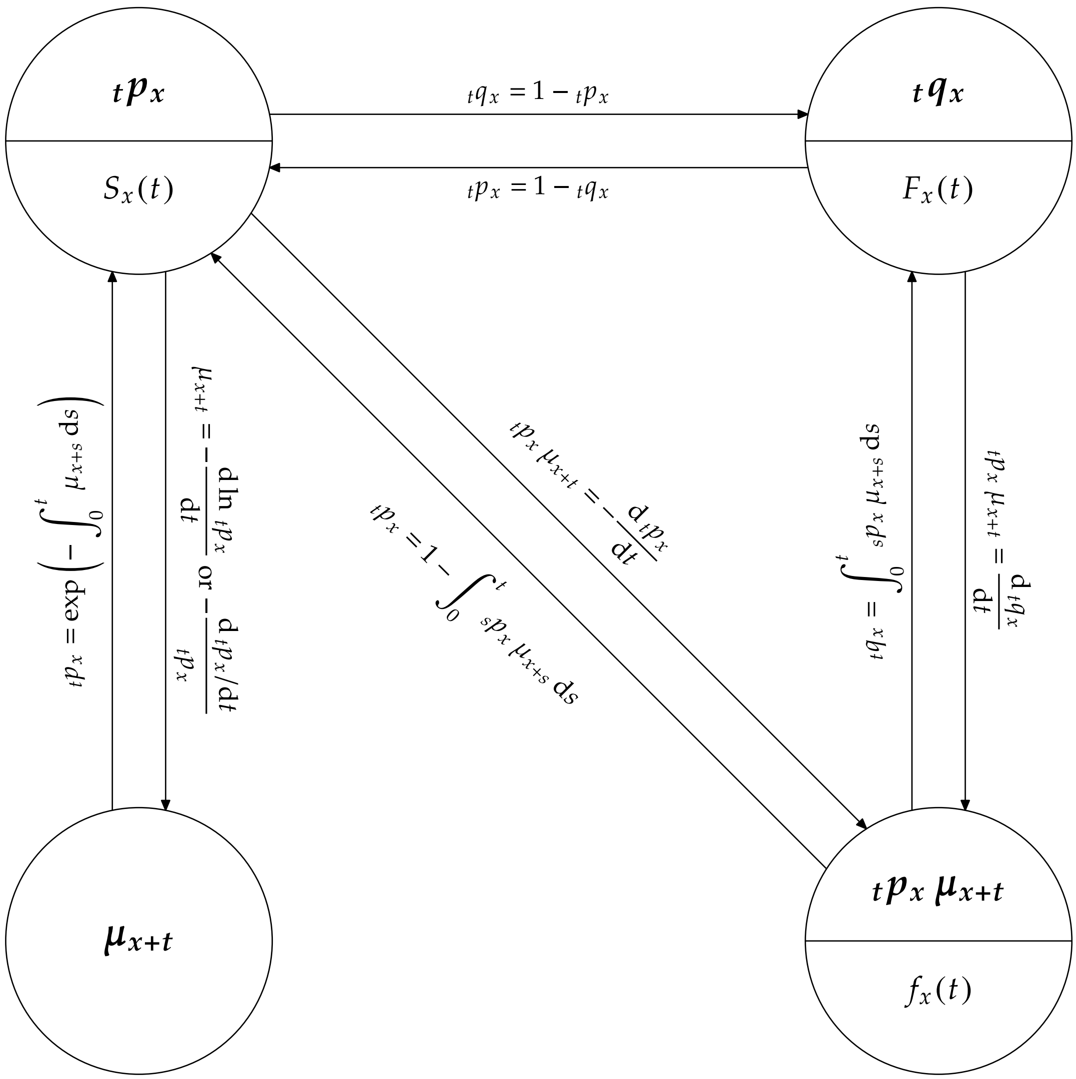


Figure 3.1: Relationships between $_tp_x$, $_tq_x$, and μ_{x+t}

Table 3.1: Formula Summary for this lesson

$$\mu_{x+t} = \frac{f_x(t)}{S_x(t)} \tag{3.1}$$

$$= \frac{\mathrm{d}\,{}_tq_x/\mathrm{d}t}{{}_tp_x} \tag{3.2}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}t}\ln S_x(t) \tag{3.3}$$

$$= -\frac{\mathrm{d}\ln {}_tp_x}{\mathrm{d}t} \tag{3.4}$$

$$S_x(t) = \exp\left(-\int_0^t \mu_{x+s}\,\mathrm{d}s\right) \tag{3.5}$$

$${}_tp_x = \exp\left(-\int_0^t \mu_{x+s}\,\mathrm{d}s\right) \tag{3.6}$$

$$= \exp\left(-\int_x^{x+t} \mu_s\,\mathrm{d}s\right) \tag{3.7}$$

$$f_x(t) = {}_tp_x\,\mu_{x+t} \tag{3.8}$$

$${}_{t|u}q_x = \int_t^{t+u} {}_sp_x\,\mu_{x+s}\,\mathrm{d}s \tag{3.9}$$

$${}_tq_x = \int_0^t {}_sp_x\,\mu_{x+s}\mathrm{d}s \tag{3.10}$$

If $\mu'_{x+s} = \mu_{x+s} + k$ for $0 \le s \le t$, then ${}_tp'_x = {}_tp_x\,e^{-kt}$. More generally, if $\mu_{x+s} = \hat{\mu}_{x+s} + \tilde{\mu}_{x+s}$ for $0 \le s \le t$, then ${}_tp_x = {}_t\hat{p}_x\,{}_t\tilde{p}_x$

If $\mu'_{x+s} = k\mu_{x+s}$ for $0 \le s \le t$, then ${}_tp'_x = \left({}_tp_x\right)^k$.

3.2. [CAS4-S88:15] (1 point) The force of mortality is

$$\mu_x = \frac{1}{100-x}$$

Calculate ${}_{10}p_{50}$.

(A) Less than 0.82
(B) At least 0.82, but less than 0.84
(C) At least 0.84, but less than 0.86
(D) At least 0.86, but less than 0.88
(E) At least 0.88

Exercises continue on the next page . . .

3.3. **[CAS4A-F97:8]** (1 point) Given that the force of mortality $\mu_x = 2x$, determine the cumulative distribution function for the random variable time until death, $F_0(x)$, the density function for that random variable, $f_0(x)$, and the survival function $S_0(x)$.

(A) $F_0(x) = 1 - e^{x^2}$ $\quad f_0(x) = 2xe^{-x^2}$ $\quad S_0(x) = e^{-x^2}$
(B) $F_0(x) = 1 - e^{-x^2}$ $\quad f_0(x) = 2e^{-x^2}$ $\quad S_0(x) = e^{x^2}$
(C) $F_0(x) = 1 - 2x$ $\quad f_0(x) = 2e^{-x^2}$ $\quad S_0(x) = e^{x^2}$
(D) $F_0(x) = 1 - e^{x^2}$ $\quad f_0(x) = xe^{-x^2}$ $\quad S_0(x) = e^{2x}$
(E) $F_0(x) = 1 - e^{-x^2}$ $\quad f_0(x) = 2xe^{-x^2}$ $\quad S_0(x) = e^{-x^2}$

3.4. **[CAS4A-F99:12]** (1 point) If $l_x = 100(k - 0.5x)^{2/3}$ and $\mu_{50} = \frac{1}{48}$, what is the value of k?

(A) Less than 40
(B) At least 40, but less than 42
(C) At least 42, but less than 44
(D) At least 44, but less than 46
(E) At least 46

3.5. The force of mortality for Kevin is $\mu_x = kx^2$. The force of mortality for Kira is $\hat{\mu}_x = 2$.

Determine the k for which ${}_5p_{10}$ is the same for Kevin and Kira.

3.6. You are given that the force of mortality is $\mu_x = 1.5(1.1^x)$, $x > 0$.

Calculate ${}_2p_1$.

3.7. You are given that

$${}_xp_0 = \frac{-x^2 - 30x + 18000}{18{,}000} \qquad 0 \le x \le 120$$

Develop an expression for μ_x valid for $0 < x < 120$.

3.8. For a standard life, ${}_5p_{45} = 0.98$. Since Boris, age 45, is recovering from surgery, he is subject to extra mortality. Therefore, the μ_x applying to Boris is increased for x between 45 and 50. The increase over the μ_x for a standard life is 0.002 at $x = 45$, decreasing in a straight line to 0 at age 50.

Calculate ${}_5p_{45}$ for Boris.

3.9. **[4-S86:26]** You are given $\mu_x = 2x/(10{,}000 - x^2)$ for $0 \le x \le 100$.

Determine q_x.

(A) $\dfrac{2x+1}{10{,}000 - x^2}$

(B) $\dfrac{4x+2}{10{,}000 - x^2}$

(C) $\dfrac{6x+3}{10{,}000 - x^2}$

(D) $\dfrac{2x+1}{29{,}999 - 3x^2 - 3x}$

(E) $\dfrac{6x+1}{29{,}999 - 3x^2 - 3x}$

Exercises continue on the next page . . .

3.10. **[4-S86:31]** A mortality table has a force of mortality μ_{x+t} and mortality rate q_x. A second mortality table has a force of mortality μ^*_{x+t} and mortality rate q^*_x.

You are given $\mu^*_{x+t} = 0.5\mu_{x+t}$ for $0 \le t \le 1$.

Calculate q^*_x.

(A) $1-\sqrt{1-q_x}$ (B) $\sqrt{q_x}$ (C) $0.5q_x$ (D) $(q_x)^2$ (E) $q_x-(q_x)^2$

3.11. **[CAS4A-S93:12]** (2 points) A mortality table for a subset of the population with better than average health is constructed by dividing the force of mortality in the standard table by 2. The probability of an 80-year-old dying within the next year is defined in the standard table as q_{80} and in the revised table it is defined as q'_{80}. In the standard table $q_{80}=0.30$.

Determine the value of q'_{80} in the revised table.

(A) Less than 0.150
(B) At least 0.150, but less than 0.155
(C) At least 0.155, but less than 0.160
(D) At least 0.160, but less than 0.165
(E) At least 0.165

3.12. **[150-S88:1]** You are given:

(i) $\hat{\mu}_{x+t} = \mu_{x+t} - k$, $0 \le t \le 1$
(ii) $\hat{q}_x = 0$, where $\hat{q}_x$ is based on the force of mortality $\hat{\mu}_{x+t}$.

Determine k.

(A) $-\ln p_x$ (B) $\ln p_x$ (C) $-\ln q_x$ (D) $\ln q_x$ (E) q_x

3.13. **[150-F88:6]** Which of the following functions can serve as a force of mortality?

I. Bc^x $\quad B>0, 0<c<1, x\ge 0$
II. $B(x+1)^{-0.5}$ $\quad B>0, x\ge 0$
III. $k(x+1)^n$ $\quad k>0, n>0, x\ge 0$

(A) I and II only (B) I and III only (C) II and III only (D) I, II and III
(E) The correct answer is not given by (A) , (B) , (C) , or (D) .

3.14. **[CAS4A-F99:13]** (2 points) Which of the following equations define valid mortality functions?

1. $\mu_x = (1+x)^{-3}$ $\quad x\ge 0$
2. $\mu_x = 0.05(1.01)^x$ $\quad x \ge 0$
3. $f_0(x) = e^{-x/2}$ $\quad x\ge 0$

(A) 1 (B) 2 (C) 1,2 (D) 1,3 (E) 2,3

3.15. **[CAS3-F04:7]** Which of the following formulas could serve as a force of mortality?

1. $\mu_x = BC^x$, $\quad B>0, C>1$
2. $\mu_x = a(b+x)^{-1}$, $\quad a>0, b>0$
3. $\mu_x = (1+x)^{-3}$, $\quad x\ge 0$

(A) 1 only (B) 2 only (C) 3 only (D) 1 and 2 only (E) 1 and 3 only

Exercises continue on the next page . . .

3.16. You are given:

(i) $f_{40}(15) = 0.0010$
(ii) ${}_{10}p_{40} = 0.96$

Calculate $f_{50}(5)$.

3.17. The probability density function for survival of a newborn is

$$f_0(t) = \frac{30t^4(100-t)}{100^6}, \quad 0 < t \le 100$$

Calculate μ_{80}.

3.18. **[150-F88:15]** You are given $F_0(x) = 1 - \frac{1}{x+1}$ for $x \ge 0$. Which of the following are true?

I. ${}_xp_0 = 1/(x+1)$

II. $\mu_{49} = 0.02$

III. ${}_{10}p_{39} = 0.80$

(A) I and II only (B) I and III only (C) II and III only (D) I, II and III
(E) The correct answer is not given by (A) , (B) , (C) , or (D) .

3.19. **[150-S90:12]** You are given

(i) $\mu_x = A + e^x$ for $x \ge 0$
(ii) ${}_{0.5}p_0 = 0.50$

Calculate A.

(A) −0.26 (B) −0.09 (C) 0.00 (D) 0.09 (E) 0.26

3.20. **[3-F00:36]** Given:

(i) $\mu_x = F + e^{2x}$, $x \ge 0$
(ii) ${}_{0.4}p_0 = 0.50$

Calculate F.

(A) −0.20 (B) −0.09 (C) 0.00 (D) 0.09 (E) 0.20

3.21. **[150-81-94:48]** You are given:

$$S_0(x) = \frac{(10-x)^2}{100} \qquad 0 \le x \le 10.$$

Calculate the difference between the force of mortality at age 1, and the probability that (1) dies before age 2.

(A) 0.007 (B) 0.010 (C) 0.012 (D) 0.016 (E) 0.024

Exercises continue on the next page . . .

3.22. **[CAS4-S86:17]** (1 point) A subgroup of lives is subject to twice the normal force of mortality. In other words,

$$\mu'_x = 2\mu_x$$

where a prime indicates the rate for the subgroup.

Express q'_x in terms of q_x.

(A) $(q_x)^2$
(B) $(q_x)^2 - 2q_x$
(C) $2q_x - (q_x)^2$
(D) $2(q_x)^2 - q_x$
(E) $2q_x + (q_x)^2$

3.23. **[CAS4A-S92:16]** (2 points) You are given that $p_{30} = 0.95$ for a standard insured with force of mortality μ_{30+t}, $0 \le t \le 1$. For a preferred insured, the force of mortality is $\mu_{30+t} - c$ for $0 \le t \le 1$.

Determine c such that the probability that (30) will die within one year is 25% lower for a preferred insured than for a standard.

(A) Less than 0.014
(B) At least 0.014, but less than 0.015
(C) At least 0.015, but less than 0.016
(D) At least 0.016, but less than 0.017
(E) At least 0.017

3.24. **[CAS4A-S96:16]** (2 points) A life table for severely disabled lives is created by modifying an existing life table by doubling the force of mortality at all ages.

In the original table, $q_{75} = 0.12$.

Calculate q_{75} in the modified table.

(A) Less than 0.21
(B) At least 0.21, but less than 0.23
(C) At least 0.23, but less than 0.25
(D) At least 0.25, but less than 0.27
(E) At least 0.27

3.25. **[CAS4A-F92:2]** (1 point) You are given $S_0(x) = e^{-x^3/12}$ for $x \ge 0$.

Determine μ_x.

(A) $-x^2/4$ (B) $1 - x^2/4$ (C) $x^2/4$ (D) $(x^2/4)e^{-x^2/12}$ (E) $-x^3/12$

3.26. You are given that ${}_tp_x = 1 - t^2/100$ for $0 < t \le 10$.

Calculate μ_{x+5}.

Exercises continue on the next page . . .

3.27. **[150-83-96:15]** You are given:

(i) $\mu_{35+t} = \mu, 0 \leq t \leq 1.$
(ii) $p_{35} = 0.985$
(iii) μ'_{35+t} is the force of mortality for (35) subject to an additional hazard, $0 \leq t \leq 1$.
(iv) $\mu'_{35+t} = \mu + c, 0 \leq t \leq 0.5$
(v) The additional force of mortality decreases uniformly from c to 0 between age 35.5 and 36.

Determine the probability that (35) subject to the additional hazard will not survive to age 36.

(A) $0.015e^{-0.25c}$
(B) $0.015e^{0.25c}$
(C) $1 - 0.985e^{-c}$
(D) $1 - 0.985e^{-0.5c}$
(E) $1 - 0.985e^{-0.75c}$

3.28. You are given the force of failure for a battery is $0.1x$, with x measured in hours of use.

Calculate the probability that the battery will last 10 hours.

3.29. **[3-S01:28]** For a population of individuals, you are given:

(i) Each individual has a constant force of mortality.
(ii) The forces of mortality are uniformly distributed over the interval $(0, 2)$.

Calculate the probability that an individual drawn at random from this population dies within one year.

(A) 0.37 (B) 0.43 (C) 0.50 (D) 0.57 (E) 0.63

3.30. **[CAS4A-F96:8]** (2 points) At birth, infants are subject to a decreasing force of mortality during the early months of life. Assume a newborn infant is subject to a force of mortality given by

$$\mu_x = \frac{1}{10 + x} \quad \text{for } x \geq 0, \text{ where } x \text{ is expressed in months.}$$

Calculate the probability that a newborn infant will survive 5 months and die in the ensuing 15 months.

(A) Less than 0.15
(B) At least 0.15, but less than 0.20
(C) At least 0.20, but less than 0.25
(D) At least 0.25, but less than 0.30
(E) At least 0.30

Exercises continue on the next page . . .

3.31. **[CAS4A-F97:11]** (2 points) You are given a life, age 30, subject to a force of mortality of

$$\mu_x = 0.02x^{0.5} \quad \text{for } 20 \le x \le 50$$

Determine the probability that this life will survive 5 years and die during the following year.

(A) Less than 0.044
(B) At least 0.044, but less than 0.052
(C) At least 0.052, but less than 0.060
(D) At least 0.060, but less than 0.068
(E) At least 0.068

3.32. **[150-F97:17]** You are given:

(i) The force of mortality is a constant, μ.
(ii) $\mu \le 1$.
(iii) ${}_{3|3}q_{33} = 0.0030$

Calculate 1000μ.

(A) 0.8 (B) 0.9 (C) 1.0 (D) 1.1 (E) 1.2

3.33. You are given that $p_x = 0.85$ for a person whose future lifetime has force of mortality μ_{x+s} for $s \le 1$. For another person, future lifetime has force of mortality $\mu'_{x+s} = 1.1\mu_{x+s} - 0.05$ for $s \le 1$.

Calculate p_x for this other person.

3.34. For a certain individual, mortality follows the Illustrative Life Table, except that the force of mortality is double the force of mortality underlying the Illustrative Life Table between ages 45 and 50.

Calculate ${}_{5|5}q_{42}$ for this individual.

3.35. **[3-S00:17]** The future lifetimes of a certain population can be modeled as follows:

(i) Each individual's future lifetime is exponentially distributed with constant hazard rate θ.
(ii) Over the population, θ is uniformly distributed over $(1, 11)$.

Calculate the probability of surviving to time 0.5, for an individual randomly selected at time 0.

(A) 0.05 (B) 0.06 (C) 0.09 (D) 0.11 (E) 0.12

3.36. **[3-F02:1]** Given: The survival function $S_0(x)$, where

$$S_0(x) = \begin{cases} 1, & 0 \le x < 1 \\ 1 - \left(e^x/100\right), & 1 \le x < 4.5 \\ 0, & 4.5 \le x \end{cases}$$

Calculate μ_4.

(A) 0.45 (B) 0.55 (C) 0.80 (D) 1.00 (E) 1.20

Exercises continue on the next page . . .

3.37. **[150-82-94:15]** You are given:

(i) $R = 1 - e^{-\int_0^1 \mu_{x+t}\,dt}$.

(ii) $S = 1 - e^{-\int_0^1 (\mu_{x+t}-k)\,dt}$.

(iii) k is a positive constant

Determine an expression for k such that $S = \frac{2}{3}R$.

(A) $\ln\bigl((1-p_x)/(1-\frac{2}{3}q_x)\bigr)$

(B) $\ln\bigl((1-\frac{2}{3}q_x)/(1-p_x)\bigr)$

(C) $\ln\bigl((1-\frac{2}{3}p_x)/(1-p_x)\bigr)$

(D) $\ln\bigl((1-q_x)/(1-\frac{2}{3}q_x)\bigr)$

(E) $\ln\bigl((1-\frac{2}{3}q_x)/(1-q_x)\bigr)$

3.38. **[3-F02:35]** You are given:

(i) μ_{x+t} is the force of mortality.

(ii) $R = 1 - e^{-\int_0^1 \mu_{x+t}\,dt}$

(iii) $S = 1 - e^{-\int_0^1 (\mu_{x+t}+k)\,dt}$

(iv) k is a constant such that $S = 0.75R$

Determine an expression for k.

(A) $\ln\bigl((1-q_x)/(1-0.75q_x)\bigr)$

(B) $\ln\bigl((1-0.75q_x)/(1-p_x)\bigr)$

(C) $\ln\bigl((1-0.75p_x)/(1-p_x)\bigr)$

(D) $\ln\bigl((1-p_x)/(1-0.75q_x)\bigr)$

(E) $\ln\bigl((1-0.75q_x)/(1-q_x)\bigr)$

3.39. **[CAS3-F04:8]** Given $S_0(x) = \bigl(1-(x/100)\bigr)^{1/2}$, for $0 \le x \le 100$, calculate the probability that a life age 36 will die between ages 51 and 64.

(A) Less than 0.15

(B) At least 0.15, but less than 0.20

(C) At least 0.20, but less than 0.25

(D) At least 0.25, but less than 0.30

(E) At least 0.30

3.40. **[SOA3-F04:4]** For a population which contains equal numbers of males and females at birth:

(i) For males, $\mu_x^m = 0.10$, $x \ge 0$

(ii) For females, $\mu_x^f = 0.08$, $x \ge 0$

Calculate q_{60} for this population.

(A) 0.076 (B) 0.081 (C) 0.086 (D) 0.091 (E) 0.096

Exercises continue on the next page . . .

3.41. [**M-S05:33**] You are given:

$$\mu_x = \begin{cases} 0.05 & 50 \le x < 60 \\ 0.04 & 60 \le x < 70 \end{cases}$$

Calculate ${}_{4|14}q_{50}$.

(A) 0.38 (B) 0.39 (C) 0.41 (D) 0.43 (E) 0.44

3.42. [**M-F05:32**] For a group of lives aged 30, containing an equal number of smokers and non-smokers, you are given:

(i) For non-smokers, $\mu_x^n = 0.08$, $x \ge 30$
(ii) For smokers, $\mu_x^s = 0.16$, $x \ge 30$

Calculate q_{80} for a life randomly selected from those surviving to age 80.

(A) 0.078 (B) 0.086 (C) 0.095 (D) 0.104 (E) 0.112

3.43. [**MLC-S07:1**] You are given:

(i) ${}_3p_{70} = 0.95$
(ii) ${}_2p_{71} = 0.96$
(iii) $\int_{71}^{75} \mu_x \, \mathrm{d}x = 0.107$

Calculate ${}_5p_{70}$.

(A) 0.85 (B) 0.86 (C) 0.87 (D) 0.88 (E) 0.89

Additional old CAS Exam 3/3L questions: S05:30, F05:11,12, S06:10,11, S08:15, F08:12, F10:1, F11:1, S12:1, F12:2

Additional old CAS Exam LC questions: S14:1, S15:2, S16:3

Solutions

3.1. We can use equation (3.6) to calculate ${}_{20}p_{70}$. Namely,

$${}_{20}p_{70} = \exp\left(-\int_0^{20} \mu_{70+s} \mathrm{d}s\right)$$

Break the integral up into two parts:

$$\int_0^5 \mu_{70+t} \, \mathrm{d}t = \int_0^5 0.01 \, \mathrm{d}t = 0.05$$
$$\int_5^{20} \mu_{70+t} \, \mathrm{d}t = \int_5^{20} 0.02 \, \mathrm{d}t = 0.3$$

It follows that ${}_{20}p_{70}$ is the product of the negative exponentiated integrals, ${}_{20}p_{70} = e^{-0.05-0.3} =$ **0.70469**.

3.2. This will be easier after the next lesson. For now, we do it from first principles, using equation (3.6):

$$\begin{aligned}
{}_{10}p_{50} &= \exp\left(-\int_0^{10} \mu_{50+s}\,\mathrm{d}s\right)\\
&= \exp\left(-\int_0^{10} \frac{\mathrm{d}s}{50-s}\right)\\
&= \exp\bigl(\ln(50-10)-\ln(50)\bigr)\\
&= \frac{40}{50} = \boxed{0.8} \qquad \textbf{(A)}
\end{aligned}$$

3.3. The choices are so poor that only **(E)** has $S_0(x) = 1 - F_0(x)$, so it is the only choice that could possibly be right, regardless of the mortality assumption.

Anyhow, the integral of $2t$ from 0 to x is x^2. We then negate and exponentiate to obtain $S_0(x) = e^{-x^2}$, which after complementing and differentiating verifies all three parts of (E).

3.4. l_x doesn't work for $x > 2k$ (it's negative), so the question is not totally accurate.

We calculate $S_0(x) = {}_xp_0 = l_x/l_0$ and then use equation (3.3).

$$\begin{aligned}
S_0(x) &= \frac{100(k-0.5x)^{2/3}}{100k^{2/3}}\\
\ln S_0(x) &= \frac{2}{3}\Bigl(\ln(k-0.5x) - \ln k\Bigr)\\
\mu_x &= -\frac{\mathrm{d}}{\mathrm{d}x}\ln S_0(x) = \frac{2}{3}\,\frac{0.5}{k-0.5x}\\
\mu_{50} &= \frac{1}{3(k-25)} = \frac{1}{48}\\
k &= \boxed{41} \qquad \textbf{(B)}
\end{aligned}$$

3.5. Since for both Kevin and Kira ${}_5p_{10}$ will be the exponential of negative an integral, it suffices to compare the integrals. For Kira, the integral of 2 from 10 to 15 is 10. For Kevin,

$$\int_{10}^{15} kx^2\mathrm{d}x = \frac{k(15^3-10^3)}{3} = 791.6667k$$

So $k = 10/791.6667 = \boxed{0.01263}$.

3.6. This is a Gompertz force of mortality, as we'll learn in the next lesson.

$$\begin{aligned}
{}_2p_1 &= \exp\left(-\int_1^3 \mu_x\mathrm{d}x\right)\\
&= \exp\left(-\int_1^3 1.5(1.1^x)\mathrm{d}x\right)\\
&= \exp\left(-\frac{1.5(1.1^3-1.1)}{\ln 1.1}\right)\\
&= e^{-3.63550} = \boxed{0.02637}
\end{aligned}$$

3.7. Notice that the numerator of ${}_xp_0$ factors as

$${}_xp_0 = \frac{-1}{18000}(x-120)(x+150)$$

Thus $\ln {}_xp_0 = -\ln 18000 + \ln(120-x) + \ln(x+150)$. Notice that $120-x$ is positive in the range $0 < x < 120$. Differentiating and negating,

$$\mu_x = \frac{1}{120-x} - \frac{1}{x+150} = -\frac{1}{x-120} - \frac{1}{x+150}$$

If you desire to put this over one denominator, you get

$$\boxed{\mu_x = -\frac{2x+30}{x^2+30x-18000}}$$

3.8. As indicated at the bottom of Table 3.1, if $\mu_{x+t} = \hat{\mu}_{x+t} + \tilde{\mu}_{x+t}$, then ${}_tp_x = {}_t\hat{p}_x\,{}_t\tilde{p}_x$. So we need to multiply 0.98 by the ${}_tp_x$ based on the increase in force.

The increase in force, which we'll call $\tilde{\mu}_x$, is a linear function with slope -0.0004 and equal to 0 at 50, so it can be written as (The constant 0.02 is selected as 50(0.0004) so that $\tilde{\mu}_{50} = 0$.)

$$\tilde{\mu}_x = 0.02 - 0.0004x$$

Then

$$\int_{45}^{50} \tilde{\mu}_x \mathrm{d}x = \int_{45}^{50} (0.02 - 0.0004x)\mathrm{d}x = 0.02(5) - 0.0004\left(\frac{50^2-45^2}{2}\right) = 0.005$$

and ${}_5\tilde{p}_{45} = e^{-0.005}$. So ${}_5p_{45} = 0.98e^{-0.005} = \boxed{0.9751}$.

One way to do the above integral quickly is to note that since the function is linear, its average value is its median, or 0.001 (half way between 0.002 and 0), and multiplying this average by the length of the interval from 45 to 50, we get 0.005.

3.9. Using equation (3.7),

$$\begin{aligned} q_x &= 1 - \exp\left(-\int_x^{x+1} \mu_t \mathrm{d}t\right) \\ &= 1 - \exp\left(-\int_x^{x+1} \frac{2t}{10{,}000-t^2}\mathrm{d}t\right) \\ &= 1 - \frac{10{,}000-(x+1)^2}{10{,}000-x^2} \\ &= \frac{(x+1)^2-x^2}{10{,}000-x^2} \\ &= \boxed{\frac{2x+1}{10{,}000-x^2}} \qquad \textbf{(A)} \end{aligned}$$

3.10. As we discussed, halving μ means taking the square root of p.

$$q_x^* = 1 - p_x^* = 1 - \sqrt{p_x} = \boxed{1-\sqrt{1-q_x}} \qquad \textbf{(A)}$$

3.11. This is the same situation as the previous exercise. $q'_{80} = 1 - \sqrt{1-0.30} = \boxed{0.1633}$. **(D)**

3.12.

$$\begin{aligned}1 = \hat{p}_x &= \exp\left(-\int_0^1 (\mu_t - k)\mathrm{d}t\right)\\ &= p_x \exp\left(-\int_0^1 (-k)\mathrm{d}t\right)\\ &= p_x\, e^k\end{aligned}$$

So $e^k = 1/p_x$, and $k = \boxed{-\ln p_x}$. **(A)**

Note: The only way q_x can be 0 is if p_x is 1, which means the integral of μ is 0 for an interval of length 1. Barring negative μ, which would be an invalid force of mortality, this can only happen if μ is 0 almost everywhere (to use the language of measure theory). So this exercise represents an artificial situation.

3.13. All of these functions are nonnegative. We need $\int_0^\infty \mu_x \mathrm{d}x = \infty$.

I. $\int_0^\infty Bc^x \mathrm{d}x = \left.\frac{Bc^x}{\ln c}\right|_0^\infty = -\frac{B}{\ln c} \neq \infty$, since $0 < c < 1$. ✗

II. $\int_0^\infty B(x+1)^{-0.5}\mathrm{d}x = 2B(x+1)^{0.5}\Big|_0^\infty = \infty$. ✓

III. $\int_0^\infty k(x+1)^n\,\mathrm{d}x = \left.\frac{k(x+1)^{n+1}}{n+1}\right|_0^\infty = \infty$. ✓

(C)

3.14.

1.

$$\int_0^\infty \mu_x\,\mathrm{d}x = \int_0^\infty (1+x)^{-3}\mathrm{d}x = -\left.\frac{1}{2(1+x)^2}\right|_0^\infty = \frac{1}{2}$$

The integral does not go to infinity. ✗

2.

$$\int_0^\infty 0.05(1.01)^x \mathrm{d}x = \frac{0.05(1.01^x)}{\ln 1.01} \to \infty \quad ✓$$

This is a Gompertz law, which we'll learn more about in subsection 4.1.1, page 64.

3.

$$\int_0^\infty f(x)\mathrm{d}x = \int_0^\infty e^{-x/2}\mathrm{d}x = 2$$

A proper density function integrated from 0 to ∞ should integrate to 1. ✗

If 3 had an extra factor $1/2$, then it would be OK.

(B)

3.15. 1 and 2 are non-negative and integrate to infinity. 3, however, has a finite integral. **(D)**

3.16. Since $f_x(t) = {}_tp_x\,\mu_{x+t}$, we have

$$\begin{aligned}{}_{15}p_{40}\,\mu_{55} &= 0.0010\\ f_{50}(5) &= {}_5p_{50}\,\mu_{55}\end{aligned}$$

However, ${}_{15}p_{40} = {}_{10}p_{40}\,{}_5p_{50}$, so

$$f_{50}(5) = \frac{{}_{15}p_{40}}{{}_{10}p_{40}}\mu_{55} = \frac{0.0010}{0.96} = \boxed{\mathbf{0.001042}}$$

You can also do this without using force of mortality, as follows:

$$f_{50}(5) = -\frac{\mathrm{d}}{\mathrm{d}t}S_{50}(t)\bigg|_{t=5} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{S_{40}(t)}{S_{40}(10)}\bigg|_{t=15} = \frac{f_{40}(15)}{{}_{10}p_{40}} = \frac{0.0010}{0.96} = \boxed{\mathbf{0.001042}}$$

3.17. We need $f_0(80)/S_0(80)$. Let's integrate f_0 to obtain S_0. Since 100 is the upper limit, the integral goes up to 100 rather than ∞.

$$\begin{aligned}S_0(80) &= \int_{80}^{100} f_0(t)\mathrm{d}t\\ &= \int_{80}^{100}\frac{30(100t^4 - t^5)\mathrm{d}t}{100^6}\\ &= \frac{30}{100^6}\left(\frac{100(100^5-80^5)}{5} - \frac{100^6-80^6}{6}\right) = 0.344640\\ f_0(80) &= \frac{30(80^4)(20)}{100^6} = 0.024576\\ \mu_{80} &= \frac{0.024576}{0.344640} = \boxed{\mathbf{0.071309}}\end{aligned}$$

3.18. ${}_xp_0 = S_0(x) = 1 - F_0(x)$, so I is true. $\ln S_0(x) = -\ln(x+1)$, and then $\mu_x = \frac{1}{x+1}$, so $\mu_{49} = 0.02$, making II true. ${}_{10}p_{39} = \frac{S_0(49)}{S_0(39)} = \frac{40}{50} = 0.80$ making III true. **(D)**

3.19. We use equation (3.7).

$$\begin{aligned}0.50 = {}_{0.5}p_0 &= \exp\left(-\int_0^{0.5}(A+e^x)\mathrm{d}x\right)\\ \ln 2 &= \int_0^{0.5}(A+e^x)\mathrm{d}x\\ 0.693147 &= 0.5A + (e^{0.5}-1)\\ 0.693147 &= 0.5A + 0.648721\\ A &= \frac{0.693147 - 0.648721}{0.5} = \boxed{\mathbf{0.08885}} \qquad \textbf{(D)}\end{aligned}$$

3.20. By equation (3.7),

$$0.50 = \exp\left(-0.4F - \int_0^{0.4} e^{2x}\mathrm{d}x\right)$$

$$\ln 0.50 = -0.4F - \frac{e^{0.8}-1}{2} = -0.4F - 0.61277$$
$$F = \frac{-0.61277 - \ln 0.50}{0.4} = \boxed{0.20094} \qquad \textbf{(E)}$$

3.21. The probability that (1) dies before age 2 is

$$q_1 = \frac{S_0(1) - S_0(2)}{S_0(1)} = \frac{0.81 - 0.64}{0.81} = 0.2099$$

The force of mortality at age 1 is

$$\mu_1 = -\frac{\mathrm{d}}{\mathrm{d}x} \ln\left(\frac{(10-x)^2}{100}\right)\bigg|_1 = \frac{2}{10-x}\bigg|_1 = \frac{2}{9} = 0.2222$$

So the difference is $0.2222 - 0.2099 = \boxed{0.0123}$ **(C)**

3.22. $p'_x = (p_x)^2$ since doubling the force of mortality squares the survival rate.

$$1 - q'_x = (1 - q_x)^2$$
$$= 1 - 2q_x + q_x^2$$
$$q'_x = 2q_x - (q_x)^2 \qquad \textbf{(C)}$$

3.23. Let primes be used for preferred insureds. Subtracting c from the force of mortality multiplies the survival probability by e^c.

$$1 - 0.75(0.05) = p'_{30} = p_{30}e^c = 0.95e^c$$
$$\frac{0.9625}{0.95} = e^c$$
$$c = \ln\left(\frac{0.9625}{0.95}\right) = \boxed{0.013072} \qquad \textbf{(A)}$$

3.24. Let primes be used for substandard insureds. Doubling the force of mortality squares the survival probability.

$$p'_{75} = (1 - q_{75})^2 = 0.7744$$
$$q'_{75} = 1 - 0.7744 = \boxed{0.2256} \qquad \textbf{(B)}$$

3.25. Using equation (3.3), we log $e^{-x^3/12}$ getting $-x^3/12$, negate the expression getting $x^3/12$, and differentiate getting $\boxed{x^2/4}$. **(C)**

3.26. Using equation (3.4),

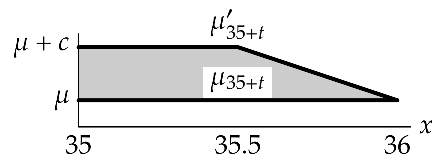
$$\mu_{x+t} = -\frac{\mathrm{d}(1 - t^2/100)/\mathrm{d}t}{1 - t^2/100} = \frac{2t}{100 - t^2}$$

so $\mu_{x+5} = \frac{10}{75} = \boxed{0.13333}$.

3.27. Let p'_{35} be the modified probability of survival. Then, by equation (3.7),

$$p'_{35} = e^{-\int_0^1 \mu'_{35+t}\mathrm{d}t} = e^{-\int_0^1 \mu_{35+t}\mathrm{d}t}e^{-\int_0^1(\mu'_{35+t}-\mu_{35+t})\mathrm{d}t} = p_{35}e^{-\int_0^1(\mu'_{35+t}-\mu_{35+t})\mathrm{d}t}$$

So $p'_x = p_x$ times e raised to the integral of negative the additional force of mortality from 0 to 1. The additional force of mortality is the difference between the two lines in the figure to the right. Its integral is the area of the shaded trapezoid, which has legs of lengths 0.5 and 1 and height c. This area is $0.5(0.5+1)c = 0.75c$. So $p'_x = 0.985e^{-0.75c}$. The answer is therefore **(E)**.



3.28. The survival probability to time x is

$$\exp\left(-\int_0^x 0.1u\,\mathrm{d}u\right) = \exp\left(-0.05x^2\right)$$

so ${}_{10}p_0 = e^{-0.05(10^2)} = e^{-5} = \boxed{\mathbf{0.006738}}$.

3.29. The law of total probability says that $P(X) = \int P(X \mid y)f(y)\mathrm{d}y$, where

$$\int_0^2 \frac{1}{2}(1-e^{-\mu})\mathrm{d}\mu = \frac{1}{2}(2+e^{-2}-1) = \boxed{\mathbf{0.5677}} \qquad \textbf{(D)}$$

3.30. We'll use monthly subscripts. Then we need ${}_5p_0$ and ${}_{20}p_0$.

$$\begin{aligned}
{}_5p_0 &= \exp\left(-\int_0^5 \frac{1}{10+x}\mathrm{d}x\right)\\
&= \exp(\ln 10 - \ln 15) = \frac{2}{3}\\
{}_{20}p_0 &= \exp\left(-\int_0^{20} \frac{1}{10+x}\mathrm{d}x\right)\\
&= \exp(\ln 10 - \ln 30) = \frac{1}{3}\\
\frac{2}{3} - \frac{1}{3} = \frac{1}{3} &= \boxed{\mathbf{0.3333}} \qquad \textbf{(E)}
\end{aligned}$$

3.31. We need ${}_5p_{30}$ and ${}_6p_{30}$.

$$\begin{aligned}
{}_5p_{30} &= \exp\left(-\int_{30}^{35} 0.02x^{0.5}\mathrm{d}x\right)\\
&= \exp\left(-\frac{0.02}{1.5}(35^{1.5}-30^{1.5})\right)\\
&= \exp\left(-\left(\frac{0.02}{1.5}\right)(42.74603)\right) = 0.565555\\
{}_6p_{30} &= \exp\left(-\left(\frac{0.02}{1.5}\right)(36^{1.5}-30^{1.5})\right)
\end{aligned}$$

$$= \exp\left(-\left(\frac{0.02}{1.5}\right)(51.68323)\right) = 0.502023$$

$$0.565555 - 0.502023 = \boxed{0.063533} \qquad \textbf{(D)}$$

3.32. Let ${}_3p$ be the probability of surviving 3 years; since μ is constant, ${}_3p$ does not vary with x. Then

$$\begin{aligned} {}_3p(1 - {}_3p) &= 0.0030 \\ {}_3p^2 - {}_3p + 0.0030 &= 0 \\ {}_3p &= \frac{1 \pm \sqrt{1 - 0.012}}{2} = 0.996991, 0.003009 \end{aligned}$$

But $\mu = -\ln p = -\frac{\ln {}_3p}{3} = 0.001005, 1.935376$. Since only the first solution is not greater than 1, the answer is $\boxed{1.005}$. **(C)**

3.33. Let p'_x be p_x for the other person. As discussed in the lesson, the adjustment to survival probabilities is

$$p'_x = e^{0.05}\left(p_x\right)^{1.1} = e^{0.05}(0.85^{1.1}) = \boxed{0.8792}$$

3.34. Doubling the force of mortality corresponds to squaring the survival rate. We need ${}_5p_{42}\,{}_5q_{47}$.

$$\begin{aligned} {}_5p_{42} &= {}_3p_{42}\,{}_2p_{45} \\ &= \left(\frac{9{,}164{,}051}{9{,}259{,}571}\right)\left(\frac{9{,}088{,}049}{9{,}164{,}051}\right)^2 = 0.973336 \\ {}_5p_{47} &= {}_3p_{47}\,{}_2p_{50} \\ &= \left(\frac{8{,}950{,}901}{9{,}088{,}049}\right)^2\left(\frac{8{,}840{,}770}{8{,}950{,}901}\right) = 0.958110 \\ {}_{5|5}q_{42} &= 0.973336(1 - 0.958110) = \boxed{0.040773} \end{aligned}$$

3.35. θ serves the same function as μ_x. The probability of survival to time 0.5 for a given individual is ${}_{0.5}p_x = e^{-0.5\theta}$. To obtain the probability of survival for a randomly chosen individual, by the Law of Total Probability, we must integrate this over the uniform distribution of the population, which has density 0.1 on the interval $[1, 11]$.

$$\begin{aligned} 0.1\int_1^{11} e^{-0.5\theta}\,\mathrm{d}\theta &= -0.2e^{-0.5\theta}\Big|_1^{11} \\ &= 0.2\left(e^{-0.5} - e^{-5.5}\right) = \boxed{0.1205} \qquad \textbf{(E)} \end{aligned}$$

3.36. Use equation (3.3).

$$\ln S_0(x) = \ln\left(1 - \left(\frac{e^x}{100}\right)\right)$$

$$\mu_x = \frac{-\mathrm{d}\ln S_0(x)}{\mathrm{d}x} = \left(\frac{e^x}{100}\right)\left(\frac{1}{1-e^x/100}\right) = \frac{e^x}{100-e^x}$$
$$\mu_4 = \frac{e^4}{100-e^4} = \boxed{1.2026} \qquad \textbf{(E)}$$

3.37. $R = 1 - p_x$, and $S = 1 - p_x e^k$, so

$$1 - p_x e^k = \frac{2}{3}(1-p_x)$$
$$e^k = \frac{1-\frac{2}{3}(1-p_x)}{p_x}$$
$$k = \ln\left(\frac{1-\frac{2}{3}(1-p_x)}{1-q_x}\right) \qquad \textbf{(E)}$$

3.38. By equation (3.7), $R = 1 - p_x = q_x$. Also,

$$1 - S = e^{-\int_0^1 (\mu_{x+t}+k)\mathrm{d}t} = e^{-\int_0^1 \mu_{x+t}\mathrm{d}t}e^{-k} = (1-R)e^{-k}$$

Substituting $S = 0.75R$:

$$1 - 0.75R = (1-R)e^{-k}$$
$$e^{-k} = \frac{1-0.75R}{1-R}$$
$$k = \ln\frac{1-R}{1-0.75R} = \boxed{\ln\frac{1-q_x}{1-0.75q_x}} \qquad \textbf{(A)}$$

3.39. Calculate the values of $S_0(x)$ that we need:

$$S_0(36) = 0.64^{1/2} = 0.8$$
$$S_0(51) = 0.49^{1/2} = 0.7$$
$$S_0(64) = 0.36^{1/2} = 0.6$$

So

$$_{15|13}q_{36} = \frac{0.7-0.6}{0.8} = \boxed{0.125} \qquad \textbf{(A)}$$

3.40. The population contains equal numbers at birth, but the numbers are not equal at age 60 due to different survivorship, so you can't simply average q_{60}^m and q_{60}^f.

Survivorship from birth to age 60 is $e^{-6} = 0.002479$ for males, $e^{-4.8} = 0.008230$ for females, so these are the relative proportions in the population.

For males, $q_{60}^m = 1 - p_{60} = 1 - e^{-0.1} = 0.095163$ and for females $q_{60}^f = 1 - e^{-0.08} = 0.076884$. Weighting these rates with the proportions in the population,

$$q_{60} = \frac{0.002479(0.095163) + 0.008230(0.076884)}{0.002479 + 0.008230} = \boxed{0.081115} \qquad \textbf{(B)}$$

As an alternative, you can calculate weighted survivorship at 60 and 61:

$$S_0(60) = 0.5(e^{-0.10(60)} + e^{-0.08(60)}) = 0.5(0.010708)$$
$$S_0(61) = 0.5(e^{-0.10(61)} + e^{-0.08(61)}) = 0.5(0.009840)$$

and then compute $q_{60} = 1 - 0.009840/0.010708 = \boxed{0.081115}$.

3.41.

$$\begin{aligned}
{}_{4|14}q_{50} &= {}_4p_{50}\,{}_{14}q_{54}\\
&= e^{-0.05(4)}\left(1-e^{-0.05(6)-0.04(8)}\right)\\
&= e^{-0.2}-e^{-0.82} = \boxed{0.378299} \qquad \textbf{(A)}
\end{aligned}$$

3.42. Calculate the probability of survival to age 80 and the probability of survival to age 81. The difference, divided by the probability of survival to age 80, is the probability that someone surviving to age 80 does not survive to age 81, which is the definition of q_{80}.

$$\begin{aligned}
{}_{51}p_{30} &= 0.5(e^{-51(0.08)}+e^{-51(0.16)}) = 0.0085967\\
{}_{50}p_{30} &= 0.5(e^{-50(0.08)}+e^{-50(0.16)}) = 0.0093256\\
q_{80} &= \frac{{}_{50}p_{30}-{}_{51}p_{30}}{{}_{50}p_{30}} = \frac{0.0093256-0.0085967}{0.0093256} = \boxed{0.078160} \qquad \textbf{(A)}
\end{aligned}$$

3.43. By equation (3.7),

$${}_4p_{71} = e^{-\int_{71}^{75}\mu_x\,dx} = e^{-0.107} = 0.898526$$

Then

$${}_5p_{70} = {}_3p_{70}\,{}_2p_{73} = 0.95\left(\frac{{}_4p_{71}}{{}_2p_{71}}\right) = \frac{0.95(0.898526)}{0.96} = \boxed{0.889166} \qquad \textbf{(E)}$$

Quiz Solutions

3-1. We need ${}_{30}p_{20} - {}_{40}p_{20}$. We must integrate μ_x from 20 to 50 and from 20 to 60.

$$\begin{aligned}
{}_tp_{20} &= \exp\left(-\int_{20}^{20+t}\mu_s\,ds\right)\\
&= \exp\left(-\int_{20}^{20+t}0.001\sqrt{s}\,ds\right)\\
&= \exp\left(-\frac{0.001}{1.5}\left((20+t)^{1.5}-20^{1.5}\right)\right)\\
{}_{30}p_{20} &= \exp\left(-\frac{0.001}{1.5}\left(50^{1.5}-20^{1.5}\right)\right)\\
&= e^{-0.176074} = 0.838556\\
{}_{40}p_{20} &= \exp\left(-\frac{0.001}{1.5}\left(60^{1.5}-20^{1.5}\right)\right)\\
&= e^{-0.250210} = 0.778637\\
{}_{30|10}q_{20} &= 0.838556-0.778637 = \boxed{0.05992}
\end{aligned}$$

Lesson 4

Survival Distributions: Mortality Laws

Reading: *Actuarial Mathematics for Life Contingent Risks* 2nd edition 2.2, 2.3, 2.7, 3.4

There are two approaches to defining $S_0(x)$, the survival function for the age at death. One is to build a life table defining the probability of survival to each integral age, and then to interpolate between integral ages. We shall learn how to interpolate in Lesson 7. The other approach is to define $S_0(x)$ as a continuous function with parameters. Using a parametric function has the advantage of capturing the distribution with only a small number of parameters, which makes the table more portable. The function may have nice properties which simplify computations. Such parametric functions are called *mortality laws*.

This lesson will discuss two types of mortality laws.

The first type of mortality law is a parametric distribution that reasonably fits human mortality or some other type of failure over a wide range of ages. It is hard to write a deep exam question using such a function, and they rarely appeared on pre-2012 exams. They will appear more often on current exams, so you should familiarize yourself with their survival functions.

The second type of mortality law is a simple parametric distribution that is quite unrealistic for human mortality, but is easy to work with. Two of the mortality laws we discuss in this section are virtually the only ones that allow easy computation of insurances and annuities in closed form without numerical methods. They appear frequently on pre-2012 exams and in the SOA sample questions. *However, they appear only occasionally on current exams.*

4.1 Mortality laws that may be used for human mortality

To get some idea of what the survival curve looks like, we will look at graphs. These graphs are imitations of the graphs in *Actuarial Mathematics for Life Contingent Risks*. However, they are based on U.S. 2006 Life Table, whereas that textbook uses three other tables. The U.S. 2006 Life Table is cut off at age 100; there is very little data above that age.

Figure 4.1 shows the survival function for three starting ages: 0, 40, and 80. These functions are related; for example $S_{40}(t) = S_0(40+t)/S_0(40)$. So each curve is a truncated, shifted, and scaled version of the previous one. $S_x(t)$ is the same as ${}_tp_x$, and if x is fixed, then $l_{x+t} = l_x\;{}_tp_x$ is a constant multiple of ${}_tp_x$, so the curves for ${}_tp_x$ and l_{x+t} look the same as the curves for $S_x(t)$.

Figure 4.2 shows the probability density function for three starting ages: 0, 40, 80. In this graph and the following ones, I used simple approximations for μ_x. The three curves are related in the same way as the three $S_x(t)$ curves are related: truncate and scale to go from one curve to another. We see that the most likely age at death is approximately 85. If we had drawn a curve for $f_x(t)$, $x > 85$, it would be monotonically decreasing. Since $f_x(t) = {}_tp_x\,\mu_{x+t}$, curves for ${}_tp_x\,\mu_{x+t}$ and $l_x\,\mu_x$ would look the same.

The most interesting graph is the one for force of mortality. Figure 4.3 graphs male and female forces of mortality. But a clearer way to see the pattern is to use a logarithmic scale, as in Figure 4.4. Characteristics of the $\ln\mu_x$ curve, based on this graph, are

1. Mortality is higher for males than for females at all ages. The lines appear to merge as age increases, and some believe male mortality may be lower than female at very high ages.

2. Mortality is very high at birth but quickly drops until about age 10.

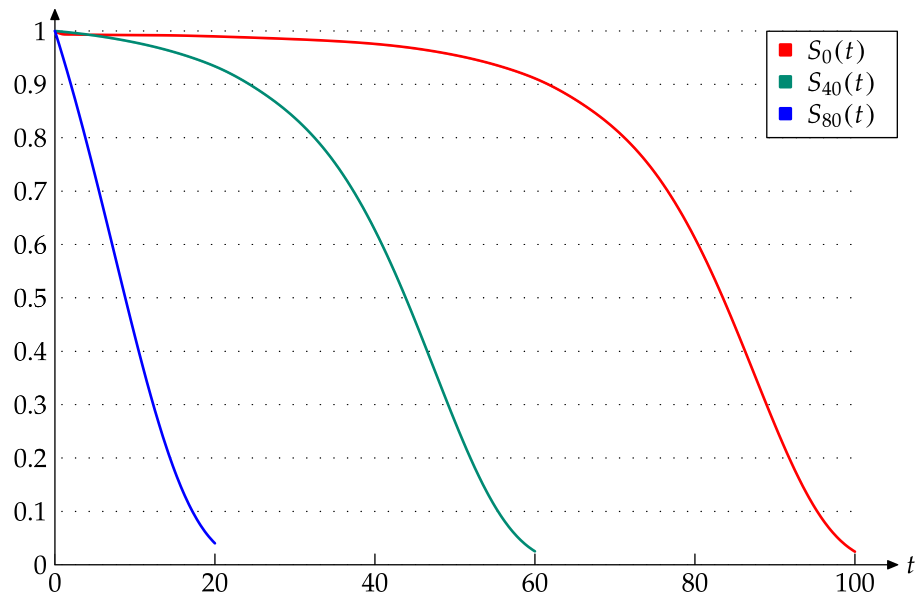


Figure 4.1: Survival function for three ages

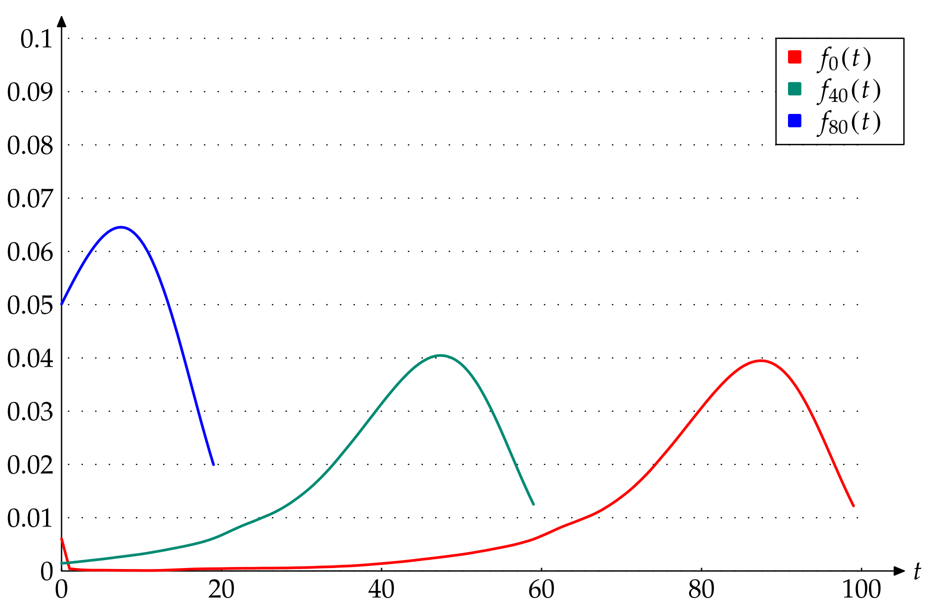


Figure 4.2: Probability density function for three ages

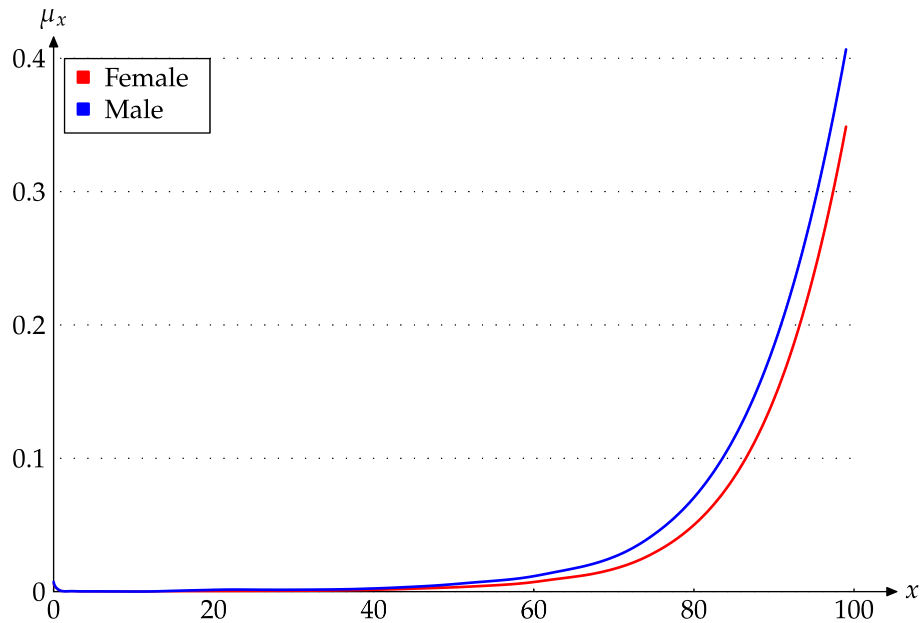


Figure 4.3: Force of mortality for males and females

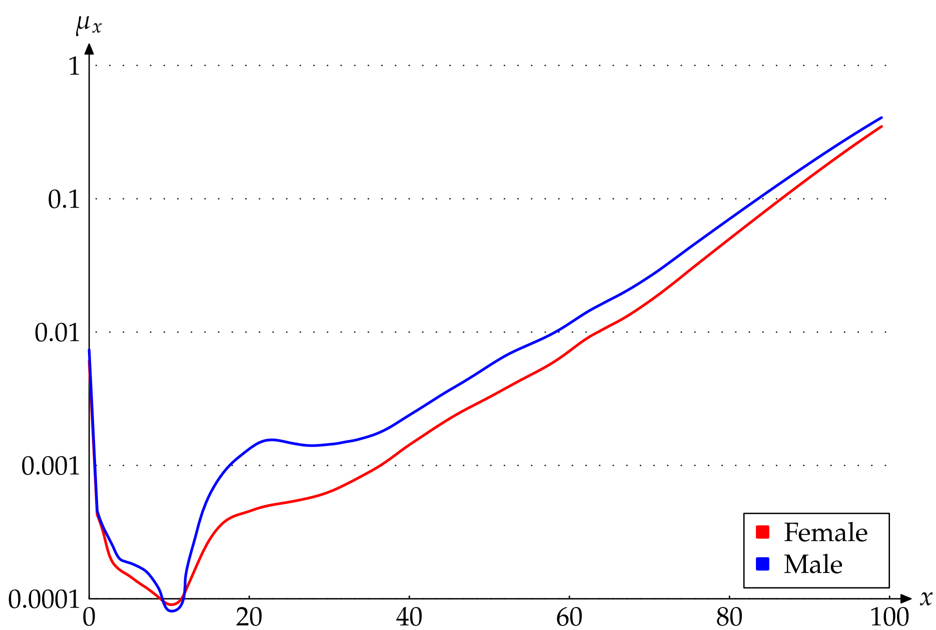


Figure 4.4: Force of mortality for males and females with logarithmic scale

3. For the male table only, there is a hump in the 20's. For both sexes, there is a rapid increase in mortality in the late teens. Both of these are caused by high accident rates, for example from youth driving.

4. Most importantly, the graphs are virtually linear starting at about age 40.

The last characteristic leads us to our first mortality law.

4.1.1 Gompertz's law

If we assume $\ln \mu_x$ is a straight line, we can solve for the parameters (by linear regression or some other means):

$$\ln \mu_x = \alpha + \beta x$$

Exponentiating, we get *Gompertz's law*:

$$\mu_x = Bc^x \tag{4.1}$$

with appropriate parameters B and $c > 1$.[1]

As we saw from Figure 4.4, this law fits ages 40 and above fairly well.

The survival function under this law is

$$\begin{aligned} {}_tp_x &= \exp\left(-\int_0^t Bc^{x+u}\,du\right) \\ &= \exp\left(-\frac{Bc^x(c^t-1)}{\ln c}\right) \end{aligned} \tag{4.2}$$

Since the law has two parameters, you can solve for all functions if you are given two values of mortality.

Example 4A Mortality for a life age 20 follows Gompertz's law. You are given $\mu_{30} = 0.001$ and $\mu_{80} = 0.15$. Determine ${}_{50}p_{20}$.

Answer: Set up two equations for ages 30 and 80.

$$\begin{aligned} \ln B + 30\ln c &= \ln 0.001 \\ \ln B + 80\ln c &= \ln 0.15 \\ 50\ln c &= \ln 150 \\ c &= e^{(\ln 150)/50} = 1.105406 \\ \ln B &= \ln 0.15 - 80\left(\frac{\ln 150}{50}\right) = -9.914136 \\ B &= 0.0000494704 \end{aligned}$$

Using formula (4.2),

$${}_{50}p_{20} = \exp\left(-\frac{0.0000494704(1.105406^{20})(1.105406^{50}-1)}{\ln 1.105406}\right) = \boxed{\mathbf{0.57937}}$$

□

[1] If $c = 1$, this is exponential mortality. If $c < 1$, then the integral $\int_0^x \mu_s\,ds$ does not approach infinity as $x \to \infty$.

4.1.2 Makeham's law

Makeham improved Gompertz's law by adding a third parameter A:

$$\mu_x = A + Bc^x \tag{4.3}$$

A represents the constant part of the force of mortality, mortality that is independent of age and is due to accidental causes. Bc^x, with $c > 1$, represents mortality resulting from deterioration and degeneration, which increases exponentially by age. Makeham's law provides a good fit for ages above 20.

As we know, adding A to μ multiplies the survival function by e^{-At}, so building on equation (4.2),

$$_tp_x = \exp\left(-At - \frac{Bc^x(c^t - 1)}{\ln c}\right) \tag{4.4}$$

Since the law has three parameters, you can solve for all functions if you are given three values of mortality. However, numerical methods may be needed to solve for the parameters.

Example 4B Mortality follows Makeham's law. You are given $\mu_{10} = 0.0014$, $\mu_{20} = 0.0024$, and $\mu_{30} = 0.0042$.

Determine $_{50}p_{20}$.

Answer: Write out the three equations for the three μ's

$$\begin{aligned}
A + Bc^{10} &= 0.0014\\
A + Bc^{20} &= 0.0024\\
A + Bc^{30} &= 0.0042\\
Bc^{10}(c^{10} - 1) &= 0.0010\\
Bc^{20}(c^{10} - 1) &= 0.0018\\
c^{10} &= 1.8\\
c &= e^{(\ln 1.8)/10} = 1.060540\\
B(1.8)(0.8) &= 0.0010\\
B &= 0.000694444\\
A + 0.000694444(1.8) &= 0.0014\\
A &= 0.00015
\end{aligned}$$

Therefore, the probability of surviving 50 years is (In the following expression, we use $c^{10} = 1.8$, so $c^{20} = (c^{10})^2 = 1.8^2$ and $c^{50} = (c^{10})^5 = 1.8^5$.)

$$_{50}p_{20} = \exp\left(-0.00015(50) - \frac{0.000694444(1.8^2)(1.8^5 - 1)}{\ln 1.060540}\right) = \boxed{0.50031}$$

□

Actuarial Mathematics for Life Contingent Risks mentions a generalization of Makeham's law: GM(r, s) (GM standing for Gompertz-Makeham) having the form

$$\mu_x = h_r^1(x) + \exp\bigl(h_s^2(x)\bigr)$$

where h_r^1 and h_s^2 are polynomials of degrees r and s respectively. Makeham's law is GM$(0, 1)$ and Gompertz's law adds the further constraint that $h_0^1 = 0$. Survival probabilities usually cannot be computed in closed form for these μ_x's, so I doubt they will appear on an exam.

4.1.3 Weibull Distribution

Occasionally an exam question will use the Weibull distribution, although it won't be called that. The exam question would just say $\mu_x = kx^n$ using some specific k and n.

For the Weibull distribution, the cumulative distribution function is

$$F(x) = 1 - e^{-(x/\theta)^\tau}$$

If you log, negate, and differentiate the survival function, you will obtain $\mu_x = \frac{\tau}{\theta}\left(\frac{x}{\theta}\right)^{\tau-1}$.

The above parametrization is the one you'll use on Exam C. The more traditional parametrization of the Weibull is $\mu_x = kx^n$, where $n > -1$ and need not be an integer. You can then derive probability functions in the usual manner.

The Weibull distribution is flexible in that n (or τ) can be set to allow mortality or failure rate to have a reasonable pattern. If $n = 0$ ($\tau = 1$), the Weibull reduces to an exponential distribution, which is *not* a reasonable mortality assumption. But if $n > 0$ (or $\tau > 1$), the force of mortality increases with age, so it becomes more reasonable. Rather than using it for human mortality, it is more commonly used for other types of failure, like machine breakdown.

After studying the constant force of mortality law in the next section, a confusion to avoid is that $\mu_x = kx$ is the force of mortality for a Weibull, not an exponential, distribution. The reason this may be confusing is that when going from force of mortality to survival rate, you integrate the force of mortality, and for an exponential, the integral of the force of mortality has the form kx.

EXAMPLE 4C You are given the force of failure for a battery is $0.1x$, with x measured in hours of use. Calculate the probability that the battery will last 10 hours.

ANSWER: The survival probability to time x is

$$\exp\left(-\int_0^x 0.1u\,du\right) = \exp\left(-0.05x^2\right)$$

so ${}_{10}p_0 = e^{-0.05(10^2)} = e^{-5} = \boxed{\mathbf{0.006738}}$. □

4.2 Mortality laws for easy computation

4.2.1 Exponential distribution, or constant force of mortality

If the force of mortality is the constant μ, the distribution of survival time is exponential. Survival probabilities are then independent of age; ${}_tp_x$ does not depend on x. The constant force of mortality μ is the reciprocal of mean survival time; in other words, a life with constant force of mortality μ has expected future lifetime $\frac{1}{\mu}$, regardless of the life's current age.

Here are the probability functions for T_x if T_x has constant force of mortality μ. Notice that none of these functions vary with x.

$$F_x(t) = 1 - e^{-\mu t}$$
$$S_x(t) = e^{-\mu t}$$
$$f_x(t) = \mu e^{-\mu t}$$
$$\mu_x = \mu \tag{4.5}$$
$$_tp_x = e^{-\mu t} \tag{4.6}$$

4.2.2 Uniform distribution

The uniform distribution on $[0, \theta]$ has a mean of $\theta/2$, which is its median and midrange. Its variance is $\theta^2/12$. It is traditional to use the letter ω to indicate the upper limit of a mortality table. For a uniform model, age at death T_0 is uniformly distributed on $(0, \omega]$.

While the uniform distribution has memory—after all, you can't live beyond ω, so the older one is, the less time until certain death—it has the following property that simplifies calculation: if age at death is uniform on $(0, \omega]$, then survival time for (x) is uniform on $(0, \omega - x]$.

Here are the probability functions for T_x if T_0 is uniform on $(0, \omega]$: After working out several problems, you should recognize the uniform distribution on sight.

$$
\begin{aligned}
F_x(t) &= \frac{t}{\omega - x} \\
S_x(t) &= \frac{\omega - x - t}{\omega - x} \\
f_x(t) &= \frac{1}{\omega - x} \\
\mu_x &= \frac{1}{\omega - x} && (4.7) \\
{}_tp_x &= \frac{\omega - x - t}{\omega - x} && (4.8) \\
{}_tq_x &= \frac{t}{\omega - x} && (4.9) \\
{}_{t|u}q_x &= \frac{u}{\omega - x} && (4.10) \\
& && (4.11)
\end{aligned}
$$

Example 4D The force of mortality for (30) is

$$\mu_x = \frac{1}{100 - x} \qquad x > 30$$

Calculate the probability of (30) dying in his 70's.

Answer: We recognize this mortality law as uniform on $(0, 100]$. For (30), remaining lifetime is uniform on $(0, 70]$. We need ${}_{40}p_{30} - {}_{50}p_{30}$.

$$
\begin{aligned}
{}_{40}p_{30} &= \frac{70 - 40}{70} = \frac{3}{7} \\
{}_{50}p_{30} &= \frac{70 - 50}{70} = \frac{2}{7}
\end{aligned}
$$

So the answer is $\frac{3}{7} - \frac{2}{7} = \boxed{\frac{1}{7}}$.

Alternatively, reason it out: mortality is uniform, and the interval $[70, 80]$ is one-seventh the size of the interval of all possible ages at death $[30, 100]$, so the probability of X being in that interval is $1/7$. □

When you work with a force of mortality like

$$\mu_x = \frac{1}{100 - x} + \frac{1}{50 - x}$$

the principle we learned in the last lesson, that the survival probability is the product of the survival probabilities corresponding to each summand in the force, is useful.

The hypothesis that mortality follows a uniform distribution is sometimes called "de Moivre's law".

4.2.3 Beta distribution

A generalized version of the uniform distribution has two parameters $\alpha > 0$ and ω and force of mortality:

$$\mu_x = \frac{\alpha}{\omega - x} \qquad 0 \le x < \omega \tag{4.12}$$

where α is a positive real number. (For a uniform distribution, $\alpha = 1$.) The force of mortality is α times the force of mortality for a uniform distribution, so survival is uniform survival raised to the α power:

$$ {}_tp_x = \left(\frac{\omega - x - t}{\omega - x}\right)^{\alpha} \tag{4.13}$$

This distribution is a special case of a beta distribution. It has the nice property that if survival time for (0) has a beta distribution, then future lifetime for (x) also has a beta distribution with the same α parameter and $\omega - x$ instead of ω as the second parameter. However, for a beta distribution, the density function is not a constant, making computations of insurance and annuity functions from basic principles impossible to do in closed form.

If T_0 follows a beta distribution with parameters ω and α, then T_x has the following characteristics for $t < \omega - x$:

$$\begin{aligned}
S_x(t) &= \left(\frac{\omega - x - t}{\omega - x}\right)^{\alpha} \\
f_x(t) &= \frac{\alpha(\omega - x - t)^{\alpha - 1}}{(\omega - x)^{\alpha}} \\
\mu_x &= \frac{\alpha}{\omega - x} \\
{}_tp_x &= \left(\frac{\omega - x - t}{\omega - x}\right)^{\alpha}
\end{aligned}$$

EXAMPLE 4E You are given that the force of mortality is

$$\mu_x = \frac{0.5}{100 - x}$$

Calculate the probability that (36) survives to age 75.

ANSWER: We recognize this as a beta with $\alpha = 0.5$ and $\omega = 100$, so the answer is

$$ {}_{39}p_{36} = \left(\frac{\omega - x - t}{\omega - x}\right)^{\alpha} = \sqrt{\frac{100 - 36 - 39}{100 - 36}} = \sqrt{\frac{25}{64}} = \boxed{\mathbf{0.625}}$$

□

Quiz 4-1 You are given that the force of mortality for (45) is

$$\mu_{45+t} = \frac{1}{270 - 3t} \qquad t < 90$$

Calculate the probability that (45) dies within the next ten years.

None of this discussion of the beta distribution, not even its name, is mentioned in the textbook. While memorizing some formulas may be useful, on a written answer question, you'd be expected to derive everything from scratch. In fact, one of the sample written answer questions provides the beta survival function and expects you to derive probabilities using it.

Exercises

Gompertz's law and Makeham's law

4.1. Mortality follows Gompertz's law with $B = 0.001$ and $c = 1.05$.

Determine x such that $l_x\,\mu_x$ is maximized.

4.2. **[CAS4-S85:17]** (1 point) Mortality follows Makeham's law, $\mu_x = A + Bc^x$.

Which of the following represents $\int_1^\infty {}_tp_x\,\mu_{x+t}\,dt$?

(A) p_x (B) q_x (C) 1 (D) 0 (E) μ_x

4.3. **[150-82-94:24]** You are given:

(i) μ_x^G denotes the force of mortality under Gompertz's law at age x where $B = 0.05$ and $c = 10^{0.04}$.
(ii) $\mu_x^W = 0.1x^n$.
(iii) $\mu_{50}^G = \mu_{50}^W$

Calculate n.

(A) 0.5 (B) 1.0 (C) 1.5 (D) 2.0 (E) 2.5

4.4. Mortality follows Gompertz's law. You are given that ${}_5p_{60} = 0.95$ and ${}_{10}p_{60} = 0.87$.

Determine ${}_{30}p_{60}$.

4.5. Mortality follows Gompertz's law. You are given that $q_{50} = 0.008$ and $q_{51} = 0.009$.

Determine μ_{50}.

4.6. For Kira, mortality follows Gompertz's law with $B = 0.00025$ and $c = 1.03$.

The force of mortality for Kevin is twice the force of mortality for Kira.

For all x, μ_x for Kevin can be expressed as μ_{x+k} for Kira.

Determine k.

4.7. Mortality follows Makeham's law. You are given that $q_{55} = 0.01$, $q_{65} = 0.02$, and $q_{75} = 0.05$.

Determine μ_{65}.

Exercises continue on the next page . . .

Table 4.1: Summary of mortality laws

Gompertz's law

$$\mu_x = Bc^x, \qquad c > 1 \tag{4.1}$$

$$_tp_x = \exp\left(-\frac{Bc^x(c^t - 1)}{\ln c}\right) \tag{4.2}$$

Makeham's law

$$\mu_x = A + Bc^x, \qquad c > 1 \tag{4.3}$$

$$_tp_x = \exp\left(-At - \frac{Bc^x(c^t - 1)}{\ln c}\right) \tag{4.4}$$

Weibull Distribution

$$\mu_x = kx^n$$

$$S_0(x) = e^{-kx^{n+1}/(n+1)}$$

Constant force of mortality

$$\mu_x = \mu \tag{4.5}$$

$$_tp_x = e^{-\mu t} \tag{4.6}$$

Uniform distribution

$$\mu_x = \frac{1}{\omega - x} \qquad 0 \le x < \omega \tag{4.7}$$

$$_tp_x = \frac{\omega - x - t}{\omega - x} \qquad 0 \le t \le \omega - x \tag{4.8}$$

$$_tq_x = \frac{t}{\omega - x} \qquad 0 \le t \le \omega - x \tag{4.9}$$

$$_{t|u}q_x = \frac{u}{\omega - x} \qquad 0 \le t + u \le \omega - x \tag{4.10}$$

Beta distribution

$$\mu_x = \frac{\alpha}{\omega - x} \qquad 0 \le x < \omega \tag{4.12}$$

$$_tp_x = \left(\frac{\omega - x - t}{\omega - x}\right)^{\alpha} \qquad 0 \le t \le \omega - x \tag{4.13}$$

Exercises continue on the next page . . .

Constant force

4.8. [**CAS4-F82:16**] In a certain population, the force of mortality is constant.

If the probability that a life age 60 will survive to age 80 is 0.10, what is the force of mortality?

(A) Less than 0.10
(B) At least 0.10, but less than 0.12
(C) At least 0.12, but less than 0.14
(D) At least 0.14, but less than 0.16
(E) At least 0.16

4.9. The probability of survival to age 80 for a newborn is 0.2. Below age 80, the force of mortality is constant.

Calculate the force of mortality below age 80.

Uniform distribution

4.10. [**CAS4-S86:16**] (1 point) Age at death is uniformly distributed on $(0, 105]$

Calculate ${}_{10|20}q_{25}$.

(A) 1/8 (B) 1/6 (C) 1/5 (D) 1/4 (E) 3/8

4.11. The force of mortality is $\mu_x = 1/(120 - x)$, $x < 120$.

Calculate ${}_{4|5}q_{30}$.

4.12. Age at death is uniformly distributed on $(0, \omega]$. You are given that $q_{10} = 1/45$.

Determine μ_{10}.

4.13. [**CAS4A-F96:5**] (2 points) A population of 20,000 lives has two subpopulations. The first subpopulation has 10,000 lives, all age 30. Age at death is uniformly distributed on $(30, 100]$. The other subpopulation has 10,000 lives, all age 40, with age at death uniformly distributed on $(40, 90]$.

Determine the expected number of people from this population of 20,000 who will die between the ages 50 and 60.

(A) Less than 3,300
(B) At least 3,300, but less than 3,500
(C) At least 3,500, but less than 3,700
(D) At least 3,700, but less than 3,900
(E) At least 3,900

4.14. Yolanda is subject to force of mortality μ_x, where

$$\mu_x = \frac{1}{100 - x} \qquad x < 100$$

Zinny is subject to a force of mortality μ'_x, where $\mu'_x = \mu_x + A$.

Zinny's mortality rate at age 15, q'_{15}, is twice Yolanda's.

Determine A.

Exercises continue on the next page . . .

4.15. The force of mortality is

$$\mu_x = \frac{1}{120-x} + \frac{1}{160-x} \qquad 0 < x < 120$$

Calculate the probability that (60) will die within the next 10 years.

Beta

4.16. **[CAS4A-S99:14]** (1 point) If $\mu_x = 1/\bigl(2(100-x)\bigr)$, calculate ${}_{40}p_{25}$.

(A) Less than 0.64
(B) At least 0.64, but less than 0.66
(C) At least 0.66, but less than 0.68
(D) At least 0.68, but less than 0.70
(E) At least 0.70

4.17. The force of mortality is

$$\mu_x = \frac{1}{3(120-x)} \qquad 0 \le x \le 120$$

Calculate ${}_{4|5}q_{30}$.

4.18. **[CAS3-F03:4]** Given:

$$\mu_x = \frac{2}{100-x}, \qquad \text{for } 0 \le x < 100$$

Calculate ${}_{10|}q_{65}$.

(A) 1/25 (B) 1/35 (C) 1/45 (D) 1/55 (E) 1/65

4.19. For (60), the probability of survival is ${}_tp_{60} = \bigl((60-t)/60\bigr)^{0.4}$
Calculate μ_{80}.

Other mortality laws

4.20. **[CAS4A-S98:14]** (1 point) Calculate the probability that a 40-year-old will survive to age 42 if the force of mortality is $\mu_x = kx^n$ with $k = 1/100$ and $n = 1$.

(A) Less than 0.20
(B) At least 0.20, but less than 0.30
(C) At least 0.30, but less than 0.40
(D) At least 0.40, but less than 0.50
(E) At least 0.50

Exercises continue on the next page . . .

Note: The following style of question is less likely to be on a current exam, since splicing has been moved to Exam C. In fact, it was removed from the list of sample questions. So you may skip it. However, you can do it even if you don't know what splicing is.

4.21. [**SOA3-F03:17**] T_0, the future lifetime of (0), has a spliced distribution.

(i) $g(t)$ follows the Illustrative Life Table.

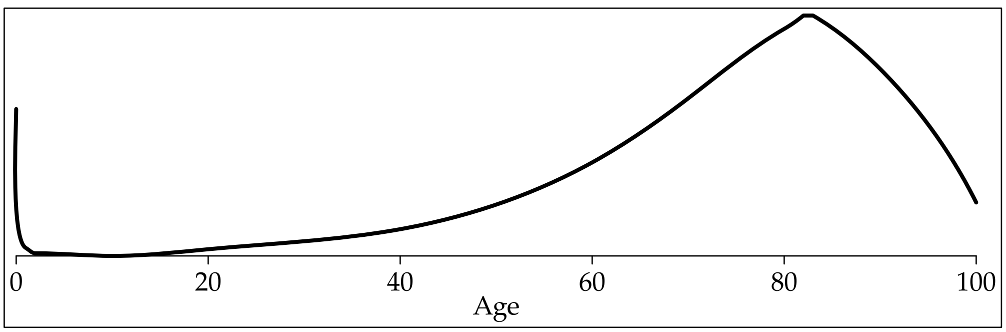
(ii) $h(t) = 0.01,\ 0 \le t \le 100$.

(iii) $f_0(t) = \begin{cases} kg(t), & 0 \le t \le 50 \\ 1.2h(t), & 50 < t \end{cases}$

Calculate ${}_{10}p_{40}$.

(A) 0.81 (B) 0.85 (C) 0.88 (D) 0.92 (E) 0.96

4.22. [**3-S01:14**] The following graph is related to current human mortality:



Which of the following functions of age does the graph most likely show?

(A) μ_x (B) $l_x\mu_x$ (C) $l_x p_x$ (D) l_x (E) l_x^2

Additional old SOA Exam MLC questions: S16:2

Additional old CAS Exam 3/3L questions: S13:2

Additional old CAS Exam LC questions: F15:1

Solutions

4.1. $l_x\,\mu_x$ is proportional to the density function, so we will calculate the density function.

$$\begin{aligned} S_0(x) &= \exp\left(-\int_0^x 0.001(1.05)^t\,\mathrm{d}t\right) \\ &= \exp\left(\frac{-0.001(1.05^x-1)}{\ln 1.05}\right) \\ f(x) &= \exp\left(\frac{-0.001(1.05^x-1)}{\ln 1.05}\right)\left(0.001(1.05)^x\right) \end{aligned}$$

For maximization purposes, we can ignore the constant 0.001. We can bring the second factor, 1.05^x, into the exponent as $\exp(x\ln 1.05)$ to get

$$\exp\left(\frac{-0.001(1.05^x-1)}{\ln 1.05}+x\ln 1.05\right)$$

and we just have to maximize the exponent. We differentiate the exponent to get

$$\begin{aligned}-0.001(1.05^x)+\ln 1.05&=0\\1.05^x&=1000\ln 1.05\\x\ln 1.05&=\ln 1000+\ln\ln 1.05\\x&=\frac{\ln 1000+\ln\ln 1.05}{\ln 1.05}=\boxed{79.6785}\end{aligned}$$

4.2. A silly question, since it has nothing to do with Makeham's law. By equation (3.9), the integral is the probability of death for (x) between time 1 and ∞, which is the same as the probability of survival until time 1. So the answer is **(A)**.

4.3. Equate the Gompertz force of mortality at 50 with the other one.

$$\begin{aligned}(0.05)\left(10^{0.04}\right)^{50}=\mu^G_{50}=\mu^W_{50}&=(0.1)(50^n)\\(0.05)(100)&=(0.1)(50^n)\\50&=50^n\\n&=\boxed{1}\qquad\textbf{(B)}\end{aligned}$$

4.4. Using a logged version of equation (4.2) in conjunction with the two probabilities we are given,

$$\begin{aligned}\frac{Bc^{60}(c^5-1)}{\ln c}&=-\ln 0.95\\\frac{Bc^{60}(c^{10}-1)}{\ln c}&=-\ln 0.87\end{aligned}$$

Dividing the first into the second,

$$\begin{aligned}c^5+1&=\frac{\ln 0.87}{\ln 0.95}=2.715015\\c&=\sqrt[5]{1.715015}=1.11392\\c^{30}-1&=24.4453\\\frac{Bc^{60}(c^{30}-1)}{\ln c}&=\left(\frac{Bc^{60}(c^5-1)}{\ln c}\right)\left(\frac{c^{30}-1}{c^5-1}\right)\\&=-\ln {}_5p_{60}\left(\frac{c^{30}-1}{c^5-1}\right)\\&=(-\ln 0.95)\left(\frac{c^{30}-1}{c^5-1}\right)=1.753641\\{}_{30}p_{60}&=e^{-1.753641}=\boxed{0.1731}\end{aligned}$$

4.5. From the logged version of equation (4.2),

$$\begin{aligned}\frac{Bc^{50}(c-1)}{\ln c} &= -\ln 0.992\\ \frac{Bc^{51}(c-1)}{\ln c} &= -\ln 0.991\end{aligned}$$

Dividing the first into the second,

$$\begin{aligned}c &= \frac{\ln 0.991}{\ln 0.992} = 1.125567\\ \mu_{50} = Bc^{50} &= \frac{(-\ln 0.992)(\ln 1.125567)}{0.125567} = \boxed{0.007566}\end{aligned}$$

4.6. The force of mortality for Kevin is $2Bc^x$. We want to express this as Bc^{x+k}.

$$\begin{aligned}2Bc^x &= Bc^{x+k} = Bc^xc^k\\ c^k &= 2\\ 1.03^k &= 2\\ k &= \frac{\ln 2}{\ln 1.03} = \boxed{23.45}\end{aligned}$$

4.7. Using the logged version of equation (4.4),

$$\begin{aligned}A + \frac{Bc^{55}(c-1)}{\ln c} &= -\ln 0.99\\ A + \frac{Bc^{65}(c-1)}{\ln c} &= -\ln 0.98\\ A + \frac{Bc^{75}(c-1)}{\ln c} &= -\ln 0.95\end{aligned}$$

Subtracting the first from the second and the second from the third,

$$\begin{aligned}\frac{Bc^{55}(c^{10}-1)(c-1)}{\ln c} &= \ln 0.99 - \ln 0.98 = 0.010152\\ \frac{Bc^{65}(c^{10}-1)(c-1)}{\ln c} &= \ln 0.98 - \ln 0.95 = 0.031091\end{aligned}$$

Dividing the first into the second,

$$\begin{aligned}c^{10} &= \frac{0.031091}{0.010152} = 3.062397\\ c &= 1.118423\\ \frac{B(1.118423^{55})(2.062397)(0.118423)}{\ln 1.118423} &= 0.010152\\ B &= \frac{(0.010152)(\ln 1.118423)}{(1.118423^{55})(2.062397)(0.118423)} = 0.0000098702\\ A &= -\ln 0.99 - \frac{0.0000098702(1.118423^{55})(0.118423)}{\ln 1.118423} = 0.005128\end{aligned}$$

The desired force of mortality is

$$\mu_{65} = 0.005128 + 0.0000098702(1.118423^{65}) = \boxed{0.01937}$$

4.8. We have $e^{-20\mu} = 0.1$, so $\mu = -(\ln 0.1)/20 = \boxed{0.11513}$. **(B)**

4.9. For a constant force of mortality, $S_0(x) = e^{-x\mu}$. So we need $e^{-80\mu} = 0.2$. Then $\mu = -\frac{\ln 0.2}{80} = \boxed{0.02012}$.

4.10. Future lifetime at (25), or T_{25}, is uniform with parameter $105 - 25 = 80$. The probability of dying over a 20 year period is therefore $\frac{20}{80} = \boxed{0.25}$. **(D)**

4.11. Future lifetime at (30), or T_{30}, is uniform with parameter $120 - 30 = 90$. The probability of dying over a 5 year period is then $5/90 = \boxed{1/18}$.

4.12. $q_{10} = 1/(\omega - 10) = 1/45$, so $\omega = 55$. Then $\mu_{10} = \frac{1}{\omega-10} = \frac{1}{55-10} = \boxed{\frac{1}{45}}$.

4.13. For the age 30 subpopulation, the uniform distribution has parameter $100 - 30 = 70$, so the probability of dying over 10 years is $1/7$. For the age 40 subpopulation, the uniform distribution has parameter $90 - 40 = 50$, so the probability of dying over 10 years is $1/5$. The total number of deaths is

$$\frac{10{,}000}{7} + \frac{10{,}000}{5} = \boxed{3429} \qquad \textbf{(B)}$$

4.14. q_{15} for Yolanda is $\frac{1}{100-15} = \frac{1}{85}$, so it is $\frac{2}{85}$ for Zinny. Letting $p'_{15} = 1 - q'_{15}$, this means $p'_{15} = \frac{83}{85}$. As we know from the previous lesson, adding a constant k to the force of mortality multiplies the survival probability by e^{-k}, so

$$\frac{83}{85} = e^{-A}\left(\frac{84}{85}\right)$$
$$\ln 83 = -A + \ln 84$$
$$A = \ln 84 - \ln 83 = \boxed{0.011976}$$

4.15. Since the force of mortality is the sum of two uniform forces, the survival probability is the product of the corresponding uniform probabilities, or

$${}_{10}p_{60} = \left(\frac{50}{60}\right)\left(\frac{90}{100}\right) = 0.75$$

so the answer is ${}_{10}q_{60} = 1 - 0.75 = \boxed{0.25}$.

Note: this exercise was set up to make it easy to solve. Suppose, instead, you were given

$$\mu_x = \frac{40}{19{,}200 - 280x + x^2} \qquad 0 < x < 120$$

Then you'd have to decompose it into partial fractions.

4.16. This is beta with $\alpha = 1/2$ and $\omega = 100$ so

$${}_{40}p_{25} = \sqrt{\frac{100 - 25 - 40}{100 - 25}} = \sqrt{35/75} = \boxed{0.6831} \qquad \textbf{(D)}$$

4.17. This is beta with $\alpha = 1/3$, $\omega = 120$, so

$$\begin{aligned} {}_4p_{30} &= \left(\frac{120-30-4}{120-30}\right)^{1/3} = 0.98496 \\ {}_9p_{30} &= \left(\frac{120-30-9}{120-30}\right)^{1/3} = 0.96549 \\ {}_{4|5}q_{30} &= 0.98496 - 0.96549 = \boxed{\mathbf{0.01947}} \end{aligned}$$

4.18. This is beta with $\alpha = 2$, $\omega = 100$, so

$$\begin{aligned} {}_{10}p_{65} &= \left(\frac{100-65-10}{100-65}\right)^{2} = \left(\frac{25}{35}\right)^2 \\ {}_{11}p_{65} &= \left(\frac{100-65-11}{100-65}\right)^{2} = \left(\frac{24}{35}\right)^2 \\ {}_{10|}q_{65} &= \frac{25^2-24^2}{35^2} = \frac{49}{1225} = \boxed{\mathbf{\frac{1}{25}}} \qquad \textbf{(A)} \end{aligned}$$

4.19. This is beta with $\omega - 60 = 60$ and $\alpha = 0.4$, so $\mu_{80} = 0.4/(60-20) = \boxed{\mathbf{0.01}}$.

4.20. We use equation (3.7).

$$\begin{aligned} {}_2p_{40} &= \exp\left(-\int_{40}^{42} \frac{1}{100} x \,\mathrm{d}x\right) \\ &= \exp\left(-0.005(42^2 - 40^2)\right) \\ &= \exp(-0.82) = \boxed{\mathbf{0.4404}} \qquad \textbf{(D)} \end{aligned}$$

4.21. Splicing the function means that at the splicing point, age 50, the distribution function must be continuous. For the uniform distribution, $h(t) = \frac{1}{100}$, so $1.2h(t) = \frac{1.2}{100}$ and the distribution function at 50 is

$$F_0(50) = 1 - \int_{50}^{100} \frac{1.2}{100} \mathrm{d}t = 0.4.$$

In the Illustrative Life Table, $l_0 = 10{,}000{,}000$ and $l_{50} = 8{,}950{,}901$, so $F_0(50) = 1 - \frac{8{,}950{,}901}{10{,}000{,}000} = 0.10491$ and $F_0(40) = 1 - \frac{9{,}313{,}166}{10{,}000{,}000} = 0.0686834$. To make $F_0(50) = 0.4$, we have to multiply f_1 by $k = \frac{0.4}{0.10491} = 3.812792$. (Multiplying f by a constant results in multiplying F by the same constant, since F is the integral of f.) But then

$$F_0(40) = 3.812792G(40) = 3.812792(0.0686834) = 0.26188$$

Therefore

$${}_{10}p_{40} = \frac{S_0(50)}{S_0(40)} = \frac{1-0.4}{1-0.26188} = \boxed{\mathbf{0.81287}} \qquad \textbf{(A)}$$

4.22. This looks like a density function, or **(B)**. μ_x would go to infinity. $l_x p_x = l_{x+1}$ and would look like l_x, both functions monotonically decreasing for all x; the same for l_x^2.

Quiz Solutions

4-1. Dividing numerator and denominator by 3,

$$\mu_{45+t} = \frac{1/3}{90 - t}$$

so this is beta (at least starting at age 45) with $\omega - 45 = 90$ and $\alpha = 1/3$. Then

$$
\begin{aligned}
{}_{10}p_{45} &= \left(\frac{90-10}{90}\right)^{1/3} = 0.961500 \\
{}_{10}q_{45} &= 1 - 0.961500 = \boxed{\mathbf{0.038500}}
\end{aligned}
$$

Lesson 5

Survival Distributions: Moments

Reading: *Actuarial Mathematics for Life Contingent Risks* 2nd edition 2.5–2.6

5.1 Complete

5.1.1 General

In International Actuarial Notation (IAN), the expected value of future lifetime for (x), $\mathbf{E}[T_x]$ is denoted by $\mathring{e}_x$. There is no symbol in International Actuarial Notation for $\operatorname{Var}(T_x)$. Expected future lifetime is also known as the *complete expectation of life* , or as *complete life expectancy*.

The expected value of a random variable X is defined as $\int_{-\infty}^{\infty} x f(x)\,dx$. Since T_x has density ${}_tp_x\,\mu_{x+t}$ and the density is positive only for $x \geq 0$, complete expectation of life is

$$\mathring{e}_x = \int_0^\infty t\,{}_tp_x\,\mu_{x+t}\,dt \tag{5.1}$$

Similarly, the second moment of T_x is

$$\mathbf{E}\left[T_x^2\right] = \int_0^\infty t^2\,{}_tp_x\,\mu_{x+t}\,dt$$

The variance of T_x is the second moment minus the mean squared.

However, under the three additional assumptions for survival functions discussed on page 20, alternative formulas that are easier to evaluate can be derived by integration by parts:

$$\mathring{e}_x = \int_0^\infty {}_tp_x\,dt \tag{5.2}$$

$$\mathbf{E}\left[T_x^2\right] = 2\int_0^\infty t\,{}_tp_x\,dt \tag{5.3}$$

$$\operatorname{Var}(T_x) = 2\int_0^\infty t\,{}_tp_x\,dt - \mathring{e}_x^2 \tag{5.4}$$

Figure 5.1 shows life expectancy for a typical mortality table.

Sometimes we're interested in the average number of years lived within the next n years. In other words, let X be the minimum of the number of years one lives and n, where n doesn't have to be an integer but usually is. We would like the expected value of X, or $\mathbf{E}\left[\min(T_x, n)\right]$. The IAN symbol for this concept is $\mathring{e}_{x:\overline{n}|}$. It is called the *n-year temporary complete life expectancy*. The formula based on the definition of expectation is

$$\mathring{e}_{x:\overline{n}|} = \mathbf{E}\left[\min(T_x, n)\right] = \int_0^n t\,{}_tp_x\,\mu_{x+t}\,dt + n\,{}_np_x \tag{5.5}$$

Those who die within n years are represented in the integral, and the survivors are represented by the second summand $n\,{}_np_x$.

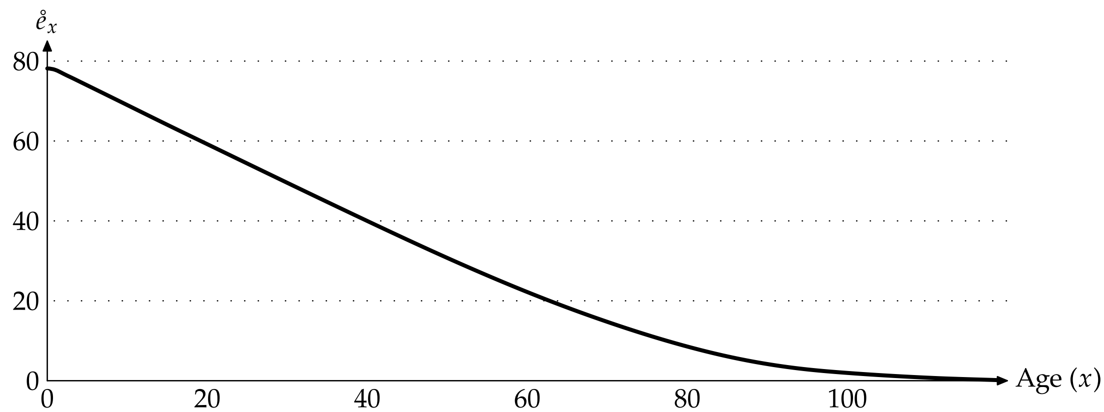


Figure 5.1: Graph of $\mathring{e}_x$

The second moment of $\min(T_x, n)$ is

$$\mathbf{E}\left[\left(\min(T_x, n)\right)^2\right] = \int_0^n t^2{}_tp_x\,\mu_{x+t}\,\mathrm{d}t + n^2{}_np_x$$

Once again, integrating by parts yields a simpler formula in which no summand is added to the integral:

$$\mathring{e}_{x:\overline{n}|} = \int_0^n {}_tp_x\,\mathrm{d}t \tag{5.6}$$

$$\mathbf{E}\left[\left(\min(T_x, n)\right)^2\right] = 2\int_0^n t\,{}_tp_x\,\mathrm{d}t \tag{5.7}$$

Example 5A For (30), you are given

$$\mu_{30+t} = \begin{cases} 0.01 & 0 < t \le 10 \\ 0.02 & t > 10 \end{cases}$$

Calculate $\mathring{e}_{30}$.

Answer: Let's calculate ${}_tp_{30}$. For $t \le 10$:

$${}_tp_{30} = \exp\left(-\int_0^t 0.01\,\mathrm{d}u\right) = e^{-0.01t}$$

For $t > 10$:

$${}_tp_{40} = \exp\left(-\int_0^t 0.02\,\mathrm{d}u\right) = e^{-0.02t}$$

$${}_tp_{30} = ({}_{10}p_{30})({}_{t-10}p_{40}) = e^{-0.1-0.02t+0.2} = e^{0.1-0.02t}$$

Using formula (5.2),

$$\begin{aligned}\mathring{e}_{30} &= \int_0^\infty {}_tp_{30}\,\mathrm{d}t \\ &= \int_0^{10} e^{-0.01t}\mathrm{d}t + \int_{10}^\infty e^{0.1-0.02t}\mathrm{d}t \\ &= 100(1-e^{-0.1}) + 50e^{0.1}(e^{-0.2}) = \boxed{\mathbf{54.7581}}\end{aligned}$$

□

Quiz 5-1 You are given

$$\mu_x = \frac{1}{2}\left(\frac{x^{-1/2}}{10-\sqrt{x}}\right) \qquad x < 100$$

Calculate $\mathring{e}_{50}$.

5.1.2 Special mortality laws

Constant force of mortality

If μ is constant, then future lifetime is exponential, so

$$\mathring{e}_x = \frac{1}{\mu} \tag{5.8}$$

$$\mathrm{Var}(T_x) = \frac{1}{\mu^2} \tag{5.9}$$

The n-year temporary complete life expectancy is:

$$\mathring{e}_{x:\overline{n|}} = \int_0^n e^{-\mu t}\,\mathrm{d}t = \frac{1-e^{-\mu n}}{\mu}$$

and the variance of temporary future lifetime can be calculated by calculating the second moment:

$$\mathbf{E}\left[\left(\min(T_x,n)\right)^2\right] = 2\int_0^n te^{-\mu t}\,\mathrm{d}t$$

$$\begin{aligned}\int_0^n te^{-\mu t}\,\mathrm{d}t &= -\left.\frac{te^{-\mu t}}{\mu}\right|_0^n + \frac{1}{\mu}\int_0^n e^{-\mu t}\,\mathrm{d}t \\ &= -\frac{ne^{-\mu n}}{\mu} + \frac{1-e^{-\mu n}}{\mu^2} \\ &= \frac{1-(1+\mu n)e^{-\mu n}}{\mu^2}\end{aligned}$$

and then calculating $\mathrm{Var}\big(\min(T_x,n)\big) = \mathbf{E}\left[\left(\min(T_x,n)\right)^2\right] - \mathring{e}^2_{x:\overline{n|}}$. We'll discuss this integral again in Subsection 12.2.1.

Temporary moments for exponentials do not appear often on exams.

Uniform and beta

If age at death is uniformly distributed on $(0,\omega]$ then remaining lifetime for (x) follows a uniform distribution. The mean is $(\omega-x)/2$, and is the same as the midrange and the median, and the variance is $(\omega-x)^2/12$. If survival follows a beta distribution with parameters α and ω, or $\mu_x = \alpha/(\omega-x)$, then

$$\mathbf{E}[T_x] = \frac{\omega-x}{\alpha+1} \tag{5.10}$$

and

$$\mathrm{Var}(T_x) = \frac{\alpha(\omega-x)^2}{(\alpha+1)^2(\alpha+2)} \tag{5.11}$$

These formulas are not in the textbook, so you'd be expected to derive them on a written answer question on the exam. For example, the expected value of a beta is

$$\begin{aligned}\int_0^{\omega-x} {}_tp_x\,\mathrm{d}t &= \int_0^{\omega-x}\left(\frac{\omega-x-t}{\omega-x}\right)^{\alpha}\mathrm{d}t\\ &= -\frac{1}{(\omega-x)^{\alpha}}\left.\frac{(\omega-x-t)^{\alpha+1}}{\alpha+1}\right|_0^{\omega-x}\\ &= \frac{\omega-x}{\alpha+1}\end{aligned}$$

Example 5B Future lifetime of (20) is subject to force of mortality $\mu_x = 0.5/(100-x)$, $x < 100$. Calculate $\mathring{e}_{20}$.

Answer: We recognize the mortality distribution as beta with $\alpha = 0.5$, $\omega = 100$. As discussed above,

$$\mathbf{E}[T_x] = \frac{\omega-x}{\alpha+1} = \frac{80}{3/2} = \boxed{53\tfrac{1}{3}}$$

□

Example 5C Kevin and Kira are age 30.

(i) Kevin's future lifetime is uniformly distributed with $\omega = 100$.

(ii) The force of mortality for Kira is

$$\mu_{30+t} = \frac{\alpha}{70-t}$$

(iii) Kira's probability of survival to age 80 is twice as high as Kevin's.

Determine Kira's expected future lifetime.

Answer: Probability Kevin survives to 80 is

$${}_{50}p_{30} = \frac{\omega-80}{\omega-x} = \frac{100-80}{100-30} = \frac{2}{7}$$

Kira's survival is beta, so ${}_tp_y$ for her is $\left(\frac{\omega-x-t}{\omega-x}\right)^{\alpha}$, and here $\omega = 100$, $x = 30$, and $t = 50$, so from (iii),

$$\begin{aligned}\left(\frac{70-50}{70}\right)^{\alpha} &= 2\left(\frac{2}{7}\right)\\ \alpha\ln(2/7) &= \ln(4/7)\\ \alpha(-1.25276) &= -0.55962\\ \alpha &= \frac{-0.55962}{-1.25276} = 0.4467\end{aligned}$$

Then since Kira's survival is beta, the expected value is

$$\mathring{e}_{30} = \frac{\omega-y}{\alpha+1} = \frac{70}{1.4467} = \boxed{48.39}$$

□

To calculate temporary life expectancy under a uniform distribution, it is not necessary to integrate. Instead, use the double expectation formula (equation (1.11) on page 7). The condition is whether $T_x > n$

or not. For those lives surviving n years, $\min(T_x, n) = n$. For those lives not surviving n years, average future lifetime is $n/2$, since future lifetime is uniformly distributed. Therefore,

$$\begin{aligned}\mathring{e}_{x:\overline{n}|} &= {}_np_x(n) + {}_nq_x(n/2)\\ &= \frac{\omega - x - n}{\omega - x}(n) + \frac{n}{\omega - x}\left(\frac{n}{2}\right)\end{aligned} \tag{5.12}$$

Just remember the logic behind this equation, not the equation itself.

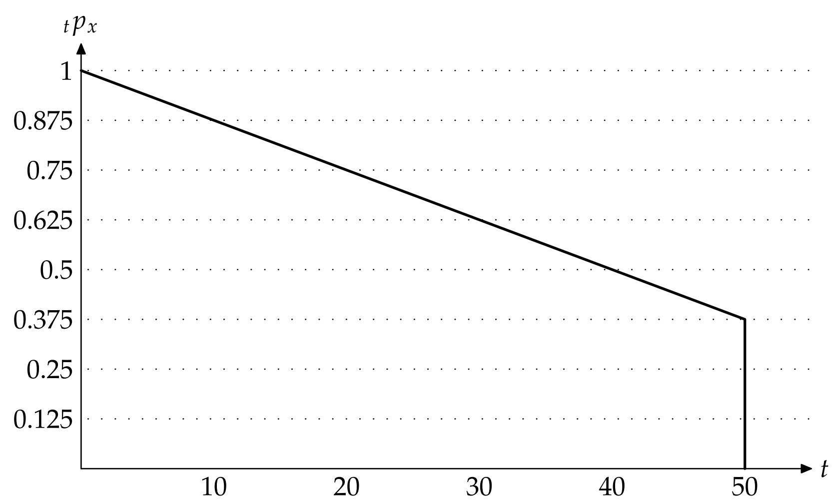
Example 5D Future lifetime of (20) is subject to force of mortality $\mu_x = \frac{1}{100-x}$, $x < 100$.
Calculate $\mathring{e}_{20:\overline{50}|}$.

Answer: The survival function is uniform with $\omega = 100$. For those who survive 50 years, $\min(T_{20}, 50) = 50$. For those who don't, the average future lifetime is 25. Therefore

$$\begin{aligned}\mathring{e}_{20:\overline{50}|} &= {}_{50}p_{20}(50) + {}_{50}q_{20}(25)\\ &= \frac{3}{8}(50) + \frac{5}{8}(25) = \frac{275}{8} = \boxed{34.375}\end{aligned}$$
□

Another method for computing temporary life expectancies under a uniform distribution is the trapezoidal rule. The idea of this rule is that to compute $\int_0^n {}_tp_x\,dt$, since the function is linear, we evaluate the area of the trapezoid bounded horizontally by the t-axis and the slanted line ${}_tp_x$ and vertically by the two lines $t = 0$ and $t = n$. The area of this trapezoid is $\mathring{e}_{x:\overline{n}|} = 0.5(1 + {}_np_x)(n)$. You can easily show that this formula is equivalent to equation (5.12).

Let's apply the trapezoidal rule to Example 5D. We need to integrate ${}_tp_{20}$ from $t = 0$ to $t = 50$. The integral is the area of this trapezoid:



At $t = 0$, ${}_0p_{20} = 1$. At $t = 50$, ${}_{50}p_{20} = 3/8$. We multiply the average of these two heights, $0.5(1 + 3/8) = 11/16$, by the width of the trapezoid, 50, to obtain $50(11/16) = 275/8 = 34.375$.

We will discuss the trapezoidal rule in more detail on page 129.

Quiz 5-2 Future lifetime for (20) follows a uniform distribution with $\omega = 120$. The n-year temporary complete life expectancy for (20) is 48.
Determine n.

We can calculate the variance of temporary future lifetime under uniform survival time using the conditional variance formula (equation (1.13) on page 9).

Example 5E (Continuation of Example 5D) Future lifetime of (20) is subject to force of mortality $\mu_x = \frac{1}{100-x}$, $x < 100$.

Calculate $\operatorname{Var}\big(\min(T_{20}, 50)\big)$.

Answer: The condition I is $T_{20} > 50$. Using this condition,

$$\begin{aligned}
\mathbf{E}[\min(T_x, 50) \mid T_{20} > 50] &= 50\\
\operatorname{Var}\big(\min(T_{20}, 50) \mid T_{20} > 50\big) &= 0\\
\mathbf{E}[\min(T_x, 50) \mid T_{20} \le 50] &= 25\\
\operatorname{Var}\big(\min(T_{20}, 50) \mid T_{20} \le 50\big) &= \frac{50^2}{12}
\end{aligned}$$

For the previous line, we used the fact that $\min(T_{20}, 50) \mid T_{20} \le 50$ is uniformly distributed on $[0, 50]$.

$$\begin{aligned}
\operatorname{Var}\big(\min(T_{20}, 50)\big) &= \mathbf{E}[0, 50^2/12] + \operatorname{Var}(50, 25)\\
&= {}_{50}q_{20}\left(\frac{50^2}{12}\right) + {}_{50}p_{20}\,{}_{50}q_{20}(50-25)^2
\end{aligned}$$

For the previous line, we used the Bernoulli shortcut (Section 1.2.1) to evaluate the variance.

$$\operatorname{Var}\big(\min(T_{20}, 50)\big) = \left(\frac{5}{8}\right)\left(\frac{2500}{12}\right) + \left(\frac{3}{8}\right)\left(\frac{5}{8}\right)(625) = \boxed{276.6927}$$

□

These methods for calculating mean and variance of future lifetime work not only for uniform mortality, but for any case in which mortality is uniformly distributed throughout the temporary period. An important case, which will be discussed in Lesson 7, is when mortality is uniform for one year. Suppose mortality for (x) is uniformly distributed over the next year. Then formula (5.12) with $n = 1$ becomes

$$\mathring{e}_{x:\overline{1}|} = p_x + 0.5q_x \tag{5.13}$$

Example 5F You are given that mortality for (x) for the next year is uniformly distributed over the year.

Demonstrate that

$$\operatorname{Var}\big(\min(T_x, 1)\big) = \frac{p_x q_x}{4} + \frac{q_x}{12}$$

Answer: The mean of a uniform random variable on $(0, 1]$ is $1/2$ and the variance is $1/12$, Using the conditional variance formula.

$$\begin{aligned}
\operatorname{Var}\big(\min(T_x, 1)\big) &= \operatorname{Var}\big(\mathbf{E}[\min(T_x, 1) \mid I]\big) + \mathbf{E}\Big[\operatorname{Var}\big(\min(T_x, 1) \mid I\big)\Big]\\
&= \operatorname{Var}(1/2, 1) + \mathbf{E}[1/12, 0]
\end{aligned}$$

In each summand, the probability of the first case is q_x and the probability of the second case is p_x. The variance of $1/2$ and 1, by the Bernoulli shortcut (page 3) is $p_x q_x (1 - 1/2)^2$, and the expected value of $1/12$ and 0 is $(1/12)q_x + (0)p_x$, so we have

$$\operatorname{Var}\big(\min(T_x, 1)\big) = \frac{p_x\, q_x}{4} + \frac{q_x}{12}$$

□

5.2 Curtate

So far we have only dealt with the random variable T_x, the complete survival time. We now introduce K_x, the random variable measuring future lifetime not counting the last fraction of a year. In other words, if time until death is 39.4, K_x would be 39. With symbols:

$$K_x = \lfloor T_x \rfloor$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. K_x is called the *curtate lifetime* (K stands for kurtate.) K_x is a discrete random variable; in fact K_x assumes integer values only.

The expected value of K_x is denoted by e_x (without the circle on the e) and is called the *curtate life expectation*. The expected value of $\min(K_x, n)$ is called the *n-year temporary curtate life expectancy* and is denoted by $e_{x:\overline{n}|}$.

From the definition of moments, formulas for the first and second moments of K_x and $\min(K_x, n)$ are:

$$e_x = \sum_{k=0}^{\infty} k\,{}_{k|}q_x \tag{5.14}$$

$$e_{x:\overline{n}|} = \sum_{k=0}^{n-1} k\,{}_{k|}q_x + n\,{}_np_x \tag{5.15}$$

$$\mathbf{E}\left[K_x^2\right] = \sum_{k=0}^{\infty} k^2\,{}_{k|}q_x \tag{5.16}$$

$$\mathbf{E}\left[\left(\min(K_x, n)\right)^2\right] = \sum_{k=0}^{n-1} k^2\,{}_{k|}q_x + n^2\,{}_np_x \tag{5.17}$$

In all of these formulas, the sum can be started at $k = 1$ instead of $k = 0$. The variance of K_x can be calculated as $\mathbf{E}\left[K_x^2\right] - e_x^2$.

As with complete expectation, better formulas can be obtained through summation by parts:

$$e_x = \sum_{k=1}^{\infty} {}_kp_x \tag{5.18}$$

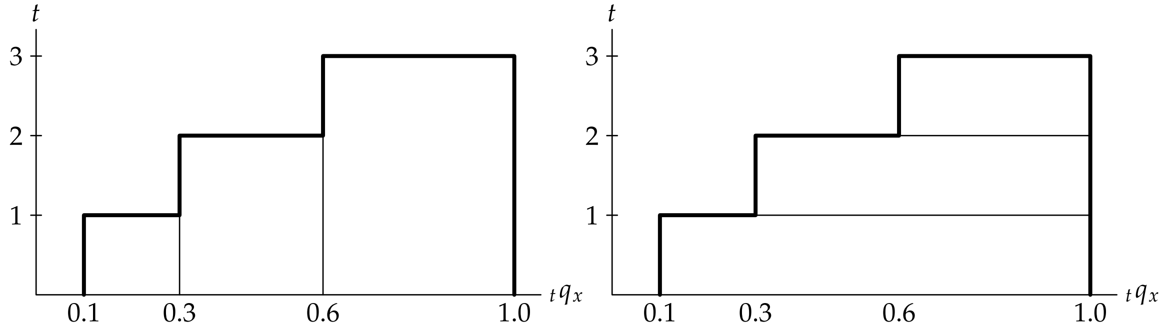
$$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x \tag{5.19}$$

$$\mathbf{E}\left[K_x^2\right] = \sum_{k=1}^{\infty} (2k-1)\,{}_kp_x \tag{5.20}$$

$$\mathbf{E}\left[\left(\min(K_x, n)\right)^2\right] = \sum_{k=1}^{n} (2k-1)\,{}_kp_x \tag{5.21}$$

To help you better understand the two alternative formulas for life expectancy, we shall demonstrate the formula graphically. Let's use a simple example. Suppose mortality follows the following distribution:

t	${}_{t-1\|}q_x$	${}_tq_x$	${}_tp_x$
1	0.1	0.1	0.9
2	0.2	0.3	0.7
3	0.3	0.6	0.4
4	0.4	1.0	0.0



(a) Using formula (5.14) (b) Using formula (5.18)

Figure 5.2: Graphic explanation of equivalence of the two formulas for e_x

To calculate e_x using formula (5.14), we add up k times the probability of living exactly (truncated to the next lowest integer) k years. This means

$$e_x = 1(0.2) + 2(0.3) + 3(0.4) = 2.0$$

This is illustrated in Figure 5.2a. The first rectangle from the left is 0.2 wide (the difference between $_2q_x$ and $_1q_x$) and 1 high, representing the first term in the sum, and so on.

To calculate e_x using formula (5.18), we add up the probabilities of living at least 1 year, 2 years, 3 years, and 4 years, or

$$e_x = 0.9 + 0.7 + 0.4 + 0 = 2.0$$

This is illustrated in Figure 5.2b. Instead of splitting the area into vertical rectangles, we now split the area into horizontal rectangles. The first rectangle from the bottom represents the probability of living at least 1 year, and so on.

The same graphical representation can be used to demonstrate the equivalence of the two formulas for *complete* expectation of life, equations (5.1) and (5.2). Instead of using intervals of 1 for t, let the size of the intervals go to zero and then the graph, instead of being a set of steps, turns into a continuous curve.

Example 5G You are given the following mortality table:

x	q_x
90	0.10
91	0.12
92	0.15
93	0.20
94	1.00

Calculate the curtate life expectancy of (90).

Answer: Calculate ${}_tp_{90}$ as cumulative products of $1 - q_x$:

$$\begin{aligned} p_{90} &= 0.90 \\ {}_2p_{90} &= (0.90)(0.88) = 0.792 \\ {}_3p_{90} &= (0.792)(0.85) = 0.6732 \\ {}_4p_{90} &= (0.6732)(0.80) = 0.53856 \\ {}_5p_{90} &= 0 \end{aligned}$$

Then $e_{90} = 0.90 + 0.792 + 0.6732 + 0.53856 = \boxed{2.90376}$. □

Quiz 5-3 You are given the following life table:

x	l_x
90	100
91	98
92	95
93	0

Calculate the variance of curtate future lifetime of (90).

Usually curtate future lifetime e_x is calculated for integer x. However, it is possible to calculate it for non-integer x; it has the same definition, the expected value of the integral number of years of future survival. In the case of temporary curtate future lifetime $e_{x:\overline{y}|}$, only the integer part of y matters; for example, $e_{32.4:\overline{5.5}|}$ is the same as $e_{32.4:\overline{5}|}$.

Example 5H You are given that $\mu_x = 0.001x$.

Calculate $e_{35.3:\overline{3}|}$.

Answer: We calculate the three survival probabilities that we need.

$$\begin{aligned} {}_tp_x &= \exp\left(-\int_0^t \mu_{x+s}\,\mathrm{d}s\right) = \exp\Bigl(-0.0005\bigl((x+t)^2 - x^2\bigr)\Bigr) \\ p_{35.3} &= e^{-0.0005(36.3^2-35.3^2)} = 0.964833 \\ {}_2p_{35.3} &= e^{-0.0005(37.3^2-35.3^2)} = 0.929973 \\ {}_3p_{35.3} &= e^{-0.0005(38.3^2-35.3^2)} = 0.895476 \\ e_{35.3:\overline{3}|} &= 0.964833 + 0.929973 + 0.895476 = \boxed{2.790282} \end{aligned}$$

□

If future lifetime is exponential, the sum ${}_tp_x$ is a geometric series.

Example 5I The force of mortality for (x) is the constant $\mu = 0.01$.

Calculate the curtate life expectancy of (x).

Answer: The probability of survival is ${}_kp_x = e^{-0.01k}$. Then

$$e_x = \sum_{k=1}^{\infty} e^{-0.01k} = \frac{e^{-0.01}}{1 - e^{-0.01}} = \boxed{99.5008}$$

□

More generally, if future lifetime is exponential with constant force μ, then

$$e_x = \sum_{k=1}^{\infty} e^{-k\mu} = \frac{e^{-\mu}}{1-e^{-\mu}} \tag{5.22}$$

Curtate life expectancy ignores the fraction of the last year of life, while complete life expectancy includes it. That fraction of the last year can range from 0 to 1, but it tends to be about 1/2 for a reasonable mortality function. Therefore, curtate future lifetime expectancy is about one half year less than complete future lifetime expectancy. If future lifetime is uniformly distributed and the limiting age ω is an integral number of years away, this relationship is exact. In symbols, the following equation holds for uniform mortality:

$$\mathring{e}_x = e_x + 0.5 \quad \text{if } \omega - x \text{ is a non-negative integer} \tag{5.23}$$

A similar equation for temporary life expectancy holds for uniform mortality:

$$\mathring{e}_{x:\overline{n|}} = e_{x:\overline{n|}} + 0.5\,{}_nq_x \quad \text{if } n \text{ is an integer and } x+n \le \omega \tag{5.24}$$

This is proved as follows: If $T_x \ge n$, then temporary curtate and complete future lifetimes are both n. If $T_x < n$, then average temporary complete future lifetime is $n/2$ and average temporary curtate future lifetime is $n/2 - 1/2$ since on the average each person dies in the middle of the year. Thus the difference between temporary curtate and complete future lifetimes is 1/2 times the probability of death before time n, or $0.5\,{}_nq_x$.

Example 5J (Same data as Example 5D) Future lifetime of (20) is subject to force of mortality $\mu_x = \frac{1}{100-x}$, $x < 100$.

1. Calculate e_{20}.
2. Calculate $e_{20:\overline{50|}}$.
3. Calculate $\mathrm{Var}(K_{20})$.

Answer: 1. For uniform mortality, $\mathring{e}_{20} = (\omega - 20)/2 = (100-20)/2 = 40$. Since $e_{20} = \mathring{e}_{20} - 0.5$, the answer is $e_{20} = \boxed{39.5}$.

2. In Example 5D we computed $\mathring{e}_{20:\overline{50|}} = 34.375$. Then

$$\begin{aligned} e_{20:\overline{50|}} &= \mathring{e}_{20:\overline{50|}} - 0.5\,{}_{50}q_{20} \\ &= 34.375 - 0.5(5/8) = \boxed{34.0625} \end{aligned}$$

3. We will use (5.16) to calculate the second moment.

$$\begin{aligned} \mathbf{E}\left[K_{20}^2\right] &= \sum_{k=0}^{79} k^2\,{}_{k|}q_{20} \\ &= \frac{1}{80}\sum_{k=0}^{79} k^2 \\ &= \frac{1}{80}\left(\frac{(79)(80)(159)}{6}\right) = 2093.5 \end{aligned}$$

because $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$. It is doubtful that an exam would expect you to know this formula.

The variance is then $2093.5 - 39.5^2 = \boxed{533.25}$. □

Quiz 5-4 For a life age 20, $\mu_{20+t} = 1/(\omega - 20 - t)$. The 24-year temporary curtate life expectancy is 20.25. Determine ω.

Exercises

5.1. A person age 70 is subject to the following force of mortality:

$$\mu_{70+t} = \begin{cases} 0.01 & t \le 5 \\ 0.02 & t > 5 \end{cases}$$

Calculate $\mathring{e}_{70}$ for this person.

5.2. **[3-F01:1]** You are given:

$$\mu_x = \begin{cases} 0.04, & 0 < x < 40 \\ 0.05, & x > 40 \end{cases}$$

Calculate $\mathring{e}_{25:\overline{25}|}$.

(A) 14.0 (B) 14.4 (C) 14.8 (D) 15.2 (E) 15.6

5.3. **[CAS4A-F97:13]** (2 points) You are given the following information about two lives:

Life	Future Lifetime Random Variable
x	Constant force of mortality $\mu_x = 0.10$
y	Constant force of mortality $\mu_y = 0.20$

Determine the ratio of (x)'s expected future lifetime between ages x and $x + 10$ to (y)'s expected future lifetime between ages y and $y + 10$.

(A) Less than 1.00
(B) At least 1.00, but less than 1.25
(C) At least 1.25, but less than 1.50
(D) At least 1.50, but less than 1.75
(E) At least 1.75

5.4. A life is subject to a constant force of mortality μ. You are given that $e_{50} = 24$.

Determine μ.

5.5. **[4-S86:22]** You are given:

(i) Age at death is uniformly distributed.
(ii) $\mathring{e}_{30} = 30$.

Calculate q_{30}.

(A) 1/30 (B) 1/60 (C) 1/61 (D) 1/62 (E) 1/70

Exercises continue on the next page . . .

Table 5.1: Formula summary for Survival Moments

Complete Future Life time

$$\mathring{e}_x = \int_0^\infty t\,{}_tp_x\,\mu_{x+t}\,dt \tag{5.1}$$

$$\mathring{e}_x = \int_0^\infty {}_tp_x\,dt \tag{5.2}$$

$$\mathring{e}_x = \frac{1}{\mu} \quad \text{for constant force of mortality}$$

$$\mathring{e}_x = \frac{\omega - x}{2} \quad \text{for uniform} \tag{5.8}$$

$$\mathring{e}_x = \frac{\omega - x}{\alpha + 1} \quad \text{for beta} \tag{5.10}$$

$$\mathbf{E}\left[\left(T_x\right)^2\right] = 2\int_0^\infty t\,{}_tp_x\,dt \tag{5.3}$$

$$\mathrm{Var}\left(T_x\right) = \frac{1}{\mu^2} \quad \text{for constant force of mortality} \tag{5.9}$$

$$\mathrm{Var}\left(T_x\right) = \frac{(\omega - x)^2}{12} \quad \text{for uniform}$$

$$\mathrm{Var}\left(T_x\right) = \frac{\alpha(\omega - x)^2}{(\alpha+1)^2(\alpha+2)} \quad \text{for beta} \tag{5.11}$$

n-year Temporary Complete Future Lifetime

$$\mathring{e}_{x:\overline{n}|} = \int_0^n t\,{}_tp_x\,\mu_{x+t}\,dt + n\,{}_np_x \tag{5.5}$$

$$\mathring{e}_{x:\overline{n}|} = \int_0^n {}_tp_x\,dt \tag{5.6}$$

$$\mathring{e}_{x:\overline{n}|} = {}_np_x(n) + {}_nq_x(n/2) \quad \text{for uniform} \tag{5.12}$$

$$\mathring{e}_{x:\overline{1}|} = p_x + 0.5q_x \quad \text{for uniform} \tag{5.13}$$

$$\mathbf{E}\left[\left(\min(T_x, n)\right)^2\right] = 2\int_0^n t\,{}_tp_x\,dt \tag{5.7}$$

Curtate Future Lifetime

$$e_x = \sum_{k=1}^\infty k\,{}_{k|}q_x \tag{5.14}$$

$$e_x = \sum_{k=1}^\infty {}_kp_x \tag{5.18}$$

$$e_x = \mathring{e}_x - 0.5 \quad \text{for uniform} \tag{5.23}$$

$$\mathbf{E}\left[\left(K_x\right)^2\right] = \sum_{k=1}^\infty (2k-1)\,{}_kp_x \tag{5.20}$$

$$\mathrm{Var}\left(K_x\right) = \mathrm{Var}\left(T_x\right) - \frac{1}{12} \quad \text{for uniform}$$

n-year Temporary Curtate Future Lifetime

$$e_{x:\overline{n}|} = \sum_{k=1}^{n-1} k\,{}_{k|}q_x + n\,{}_np_x \tag{5.15}$$

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x \tag{5.19}$$

$$e_{x:\overline{n}|} = \mathring{e}_{x:\overline{n}|} - 0.5\,{}_nq_x \quad \text{for uniform} \tag{5.24}$$

$$\mathbf{E}\left[\left(\min(K_x, n)\right)^2\right] = \sum_{k=1}^{n} (2k-1)\,{}_kp_x \tag{5.21}$$

Exercises continue on the next page . . .

5.6. [**CAS3-F03:5**] Given:

(i) Age at death is uniformly distributed.
(ii) $\mathring{e}_{20} = 30$

Calculate q_{20}.

(A) 1/60 (B) 1/70 (C) 1/80 (D) 1/90 (E) 1/100

5.7. [**150-F97:1**] For the current type of refrigerator, you are given:

(i) $S_0(x) = 1 - x/\omega$, $0 \le x \le \omega$
(ii) $\mathring{e}_0 = 20$

For a proposed new type, with the same ω, the new survival function is:

$$S_0^*(x) = \begin{cases} 1, & 0 \le x \le 5 \\ (\omega - x)/(\omega - 5), & 5 < x \le \omega \end{cases}$$

Calculate the increase in life expectancy at time 0.

(A) 2.25 (B) 2.50 (C) 2.75 (D) 3.00 (E) 3.25

5.8. [**C3 Sample:6**] You are given:

- Hens lay an average of 30 eggs each month until death.
- The survival function for hens is $S_0(m) = 1 - \frac{m}{72}$, $0 \le m \le 72$, where m is in months.
- 100 hens have survived to age 12 months.

Calculate the expected total number of eggs to be laid by these 100 hens in their remaining lifetimes.

(A) 900 (B) 3000 (C) 9000 (D) 30,000 (E) 90,000

5.9. [**3-S01:1**] For a given life age 30, it is estimated that an impact of a medical breakthrough will be an increase of 4 years in $\mathring{e}_{30}$, the complete expectation of life.

Prior to the medical breakthrough, $S_0(t) = 1 - \dfrac{t}{100}$, $0 \le t \le 100$.

After the medical breakthrough, $S_0(t) = 1 - \dfrac{t}{\omega}$, $0 \le t \le \omega$.

Calculate ω.

(A) 104 (B) 105 (C) 106 (D) 107 (E) 108

5.10. Future lifetime for (20) follows a uniform distribution. You are given:

(i) $\mathring{e}_{20:\overline{2n}|} = 25$
(ii) $\mathring{e}_{20:\overline{4n}|} = 40$
(iii) $n < (\omega - 20)/4$

Determine $\mathring{e}_{20:\overline{3n}|}$.

Exercises continue on the next page . . .

5.11. **[150-S88:11]** You are given:

(i) $l_x = (100-x)^{0.5}$, $0 \le x \le 100$
(ii) $\mathring{e}_{36:\overline{28}|} = 24.67$

Calculate $\int_0^{28} t\ {}_tp_{36}\ \mu_{36+t}\,\mathrm{d}t$.

(A) 3.67 (B) 5.00 (C) 11.33 (D) 19.67 (E) 24.67

5.12. **[150-S91:23]** A survival function, $S_0(x)$, is defined as follows:

$$S_0(x) = \left(1 - \frac{x}{\omega}\right)^r \qquad 0 \le x \le \omega, r > 0$$

For age y, $0 \le y < \omega$, you are given:

(i) $\mu_y = 0.1$
(ii) $\mathring{e}_y = 8.75$

Calculate r.

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

5.13. **[150-81-94:8]** You are given:

$$S_0(x) = \left(1 - \frac{x}{\omega}\right)^\alpha, \quad 0 \le x < \omega, \text{ where } \alpha \text{ is a positive constant}$$

Calculate $\mu_x \cdot \mathring{e}_x$.

(A) $\dfrac{\alpha}{\alpha+1}$ (B) $\dfrac{\alpha\omega}{\alpha+1}$ (C) $\dfrac{\alpha^2}{\alpha+1}$ (D) $\dfrac{\alpha^2}{\omega-x}$ (E) $\dfrac{\alpha(\omega-x)}{(\alpha+1)\omega}$

5.14. **[150-83-96:25]** You are given:

(i) $S_0(x) = \dfrac{\sqrt{k^2-x}}{k}$, $0 \le x \le k^2$, $k > 0$
(ii) $\mathring{e}_{40} = 2\mathring{e}_{80}$.

Calculate $\mathring{e}_{60}$.

(A) 10 (B) 20 (C) 30 (D) 40 (E) 50

5.15. **[150-S98:25]** You are given:

(i) $S_0(x) = \dfrac{(k^3-x)^{1/3}}{k}$, $0 \le x \le k^3$, $k > 0$
(ii) $\mathring{e}_{40} = 2\mathring{e}_{80}$

Calculate $\mathring{e}_{60}$.

(A) 40 (B) 45 (C) 50 (D) 55 (E) 60

Exercises continue on the next page . . .

5.16. **[3-F00:31]** For an industry-wide study of patients admitted to hospitals for treatment of cardiovascular illness in 1998, you are given:

(i)

Duration In Days	Number of Patients Remaining Hospitalized
0	4,386,000
5	1,461,554
10	486,739
15	161,805
20	53,488
25	17,384
30	5,349
35	1,337
40	0

(ii) Discharges from the hospital are uniformly distributed between durations shown in the table.

Calculate the mean residual time remaining hospitalized, in days, for a patient who has been hospitalized for 21 days.

(A) 4.4 (B) 4.9 (C) 5.3 (D) 5.8 (E) 6.3

5.17. **[SOA3-F04:24]** The future lifetime of (0) follows

$$S_0(x) = \left(\frac{50}{50+x}\right)^3 \qquad x > 0$$

Calculate $\mathring{e}_{20}$.

(A) 5 (B) 15 (C) 25 (D) 35 (E) 45

5.18. A life is subject to force of mortality

$$\mu_x = \frac{1}{200-x} + \frac{1}{100-x} \qquad x < 100$$

Calculate the complete life expectancy at age 0 for this life.

5.19. For a life whose survival function is

$$S_0(x) = \frac{\omega - x}{\omega}$$

you are given that $e_{10:\overline{20}|} = 18$.

Determine ω.

Exercises continue on the next page . . .

5.20. **[CAS4-S90:2]** (1 point) Mortality follows $l_x = 1000(1 - \frac{x}{100})$ for $0 \le x \le 100$.

Calculate e_{90}

(A) Less than 4.2
(B) At least 4.2, but less than 4.4
(C) At least 4.4, but less than 4.6
(D) At least 4.6, but less than 4.8
(E) At least 4.8

5.21. The force of mortality for a life is

$$\mu_x = \begin{cases} 0.01 & 0 \le x < 50 \\ 0.01 + \frac{1}{100-x} & 50 \le x < 100 \end{cases}$$

Calculate $\mathring{e}_{40}$.

5.22. Light bulbs have the following distribution for the amount of time until burning out:

Time t in hours	$F(t)$
0–4800	0
4800–6000	$(t-4800)/1200$
6000	1

Each bulb uses 0.015 kilowatt-hours of electricity per hour.

Calculate the expected number of kilowatt-hours used by 50 bulbs in their first 5000 hours.

5.23. **[CAS4-S90:11]** (2 points) You are given:

(i) Future survival time is uniformly distributed.
(ii) $\mathring{e}_{20} = 45$

Calculate the variance of the future lifetime of a person age 20, $\mathrm{Var}(T_{20})$, to the nearest integer.

(A) 108 (B) 275 (C) 350 (D) 675 (E) 700

5.24. **[150-F87:11]** You are given:

(i) Age at death is uniformly distributed.
(ii) $\mathrm{Var}(T_{50}) = 192$.

Calculate ω.

(A) 98 (B) 100 (C) 107 (D) 110 (E) 114

Exercises continue on the next page . . .

5.25. [**CAS4A-S94:3**] (2 points) You are given:

$$S_0(x) = 1 - \frac{x}{100} \text{ for } 0 \le x \le 100$$

Determine $\mathrm{Var}(T_{20})$.

(A) Less than 600
(B) At least 600, but less than 800
(C) At least 800, but less than 1,000
(D) At least 1,000, but less than 1,200
(E) At least 1,200

5.26. For a life with survival function

$$S_0(x) = \frac{\omega - x}{\omega} \quad x \le \omega$$

you are given

(i) $\omega > 40$
(ii) For a life age 30, the variance of the number of years lived between 30 and 40 is 3.5755.

Determine ω.

5.27. [**150-S89:A1**] You are given the survival function $S_0(x) = e^{-0.05x}$ for $x \ge 0$. Calculate each of the following.

(i) ${}_{5|10}q_{30}$
(ii) $F_0(30)$
(iii) $\mathring{e}_{30}$
(iv) $\mathrm{Var}(T_{30})$

5.28. [**150-F89:2**] You are given:

(i) Age at death is uniformly distributed.
(ii) $\mathrm{Var}(T_{15}) = 675$

Calculate $\mathring{e}_{25}$.

(A) 37.5 (B) 40.0 (C) 42.5 (D) 45.0 (E) 47.5

5.29. [**3-S00:1**] Given:

(i) $\mathring{e}_0 = 25$
(ii) $l_x = \omega - x,\ 0 \le x \le \omega$
(iii) T_x is the future lifetime random variable.

Calculate $\mathrm{Var}(T_{10})$.

(A) 65 (B) 93 (C) 133 (D) 178 (E) 333

5.30. Mortality follows the Illustrative Life Table.

Calculate the variance of the number of complete years lived by (67) before reaching age 70.

Exercises continue on the next page . . .

5.31. You are given that $q_x = 0.3$, $q_{x+1} = 0.5$, $q_{x+2} = 0.7$, and $q_{x+3} = 1$.

Calculate the variance of curtate future lifetime, $\operatorname{Var}(K_x)$.

5.32. You are given:

(i) The force of mortality is the constant μ.

(ii) $e_{35} = 49$.

Calculate the revised value of e_{35} if the force of mortality is changed to the constant $\mu + 0.01$.

5.33. [**SOA3-F03:28**] For (x):

(i) K is the curtate future lifetime random variable.

(ii) $q_{x+k} = 0.1(k+1), \qquad k = 0, 1, 2, \ldots, 9$

(iii) $X = \min(K, 3)$

Calculate $\operatorname{Var}(X)$.

(A) 1.1 (B) 1.2 (C) 1.3 (D) 1.4 (E) 1.5

5.34. [**M-S05:21**] You are given:

(i) $\mathring{e}_{30:\overline{40}|} = 27.692$

(ii) $S_0(x) = 1 - \dfrac{x}{\omega}, \qquad 0 \le x \le \omega$

(iii) T_x is the future lifetime variable for (x).

Calculate $\operatorname{Var}(T_{30})$.

(A) 332 (B) 352 (C) 372 (D) 392 (E) 412

5.35. The force of mortality for a life is

$$\mu_x = \frac{1}{0.5(110 - x)} \quad 0 \le x < 110$$

You are given

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Calculate the curtate expectation of life for (30).

(A) 26.17 (B) 26.42 (C) 26.67 (D) 26.92 (E) 27.17

Exercises continue on the next page . . .

5.36. [**M-F05:13**] The actuarial department for the SharpPoint Corporation models the lifetime pencil sharpeners from purchase using $S_0(t) = (1 - t/\omega)^\alpha$, for $\alpha > 0$ and $0 \le x \le \omega$.

A senior actuary examining mortality tables for pencil sharpeners has determined that the original value of α must change. You are given:

(i) The new complete expectation of life at purchase is half what it was originally.
(ii) The new force of mortality for pencil sharpeners is 2.25 times the previous force of mortality for all durations.
(iii) ω remains the same.

Calculate the original value of α.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

5.37. [**M-F05:14**] You are given:

(i) T is the future lifetime random variable.
(ii) $\mu_x = \mu$, $x \ge 0$
(iii) $\mathrm{Var}(T) = 100$
(iv) $X = \min(T, 10)$

Calculate $\mathbf{E}[X]$.

(A) 2.6 (B) 5.4 (C) 6.3 (D) 9.5 (E) 10.0

5.38. [**M-F06:2**] You are given the survival function

$$S_0(t) = 1 - (0.01t)^2 \qquad 0 \le t < 100$$

Calculate $\mathring{e}_{30:\overline{50}|}$, the 50-year temporary complete expectation of life for (30).

(A) 27 (B) 30 (C) 34 (D) 37 (E) 41

5.39. [**M-F06:23**] You are given 3 mortality assumptions:

(i) Illustrative Life Table (ILT)
(ii) Constant force model (CF), where $S_0(t) = e^{-\mu t}$, $x \ge 0$
(iii) DeMoivre model (DM), where $S_0(t) = 1 - \dfrac{t}{\omega}$, $0 \le t \le \omega$, $\omega \ge 72$.

For the constant force and DeMoivre models, ${}_2p_{70}$ is the same as for the Illustrative Life Table.

Rank $e_{70:\overline{2}|}$ for these 3 models.

(A) ILT < CF < DM
(B) ILT < DM < CF
(C) CF < DM < ILT
(D) DM < CF < ILT
(E) DM < ILT < CF

5.40. **[MLC-S07:21]** You are given the following information about a new model for buildings with limiting age ω.

(i) The expected number of buildings surviving at age x will be $l_x = (\omega - x)^\alpha$, $x < \omega$.
(ii) The new model predicts a $33\frac{1}{3}\%$ higher complete life expectancy (over the previous uniform model with the same ω) for buildings aged 30.
(iii) The complete life expectancy for buildings aged 60 under the new model is 20 years.

Calculate the complete life expectancy under the previous uniform model for buildings aged 70.

(A) 8 (B) 10 (C) 12 (D) 14 (E) 16

5.41. You are given the following mortality table for 2011, along with reduction factors:

Age x	70	71	72	73	74
q_x	0.010	0.012	0.015	0.02	0.03
Reduction factor	0.98	0.982	0.984	0.986	0.988

Calculate the excess of the 3-year temporary curtate life expectancy for a person age 71 in 2016 over the 3-year temporary curtate life expectancy for a person age 71 in 2011.

Additional old SOA Exam MLC questions: F12:3, F15:B1(b)

Additional old CAS Exam 3/3L questions: F05:10, S07:6, S08:13, S09:1, F09:2, S10:3, S11:1, F11:2, F12:1, F13:2

Additional old CAS Exam LC questions: F14:2,3, F15:2, S16:1

Written answer sample questions: 1

Solutions

5.1. Use equation (5.2).

$$\mathring{e}_{70} = \int_0^\infty {}_tp_{70}\,\mathrm{d}t$$

To calculate ${}_tp_{70}$, we consider two cases: $t \le 5$ and $t > 5$. For $t \le 5$, ${}_tp_{70} = e^{-0.01t}$. For $t > 5$, ${}_tp_{70}$ is the exponential of the negative integral of μ_{70+t}. That integral is $e^{-0.01t}$ for $t \le 5$ and $e^{-0.05-0.02(t-5)}$ for $t > 5$. (See exercise 3.1 for an example of how to calculate ${}_{20}p_{70}$.) So

$$\begin{aligned}
\mathring{e}_{70} &= \int_0^5 e^{-0.01t}\,\mathrm{d}t + \int_5^\infty e^{-0.05-0.02(t-5)}\mathrm{d}t \\
&= 100\left(1 - e^{-0.05}\right) + e^{-0.05}\int_5^\infty e^{-0.02(t-5)}\mathrm{d}t \\
&= 100\left(1 - e^{-0.05}\right) + e^{-0.05}(50) \\
&= 4.8771 + 47.5615 = \boxed{\mathbf{52.4386}}
\end{aligned}$$

5.2. We want to use equation (5.6), so we have to compute ${}_tp_{25}$.

$$\begin{aligned}
{}_tp_{25} &= e^{-\int_0^t \mu_{25+u}\,\mathrm{d}u} \\
&= e^{-0.04t} \qquad t \le 15
\end{aligned}$$

$$= e^{-0.04(15)} e^{-\int_{15}^{t} \mu_{25+u}\,du}$$
$$= e^{-0.6-0.05(t-15)} \qquad t > 15$$

Now we calculate $\mathring{e}_{25:\overline{25}|}$.

$$\begin{aligned}
\mathring{e}_{25:\overline{25}|} &= \int_0^{15} {}_tp_{25}\,dt + \int_{15}^{25} {}_tp_{25}\,dt \\
&= \int_0^{15} e^{-0.04t}dt + \int_{15}^{25} e^{-0.6-0.05(t-15)}dt \\
&= 25(1-e^{-0.6}) + 20e^{-0.6}(1-e^{-0.5}) \\
&= 25 - 5e^{-0.6} - 20e^{-1.1} \\
&= 25 - 5(0.548812) - 20(0.332871) = \boxed{\mathbf{15.598520}} \qquad \textbf{(E)}
\end{aligned}$$

5.3. We will use equation (5.6). For (x),

$$\begin{aligned}
\mathring{e}_{x:\overline{10}|} &= \int_0^{10} e^{-0.1t}dt \\
&= 10(1-e^{-1}) = 6.3212
\end{aligned}$$

For (y),

$$\begin{aligned}
\mathring{e}_{y:\overline{10}|} &= \int_0^{10} e^{-0.2t}dt \\
&= 5(1-e^{-2}) = 4.3233
\end{aligned}$$

The ratio is $6.3212/4.3233 = \boxed{\mathbf{1.4621}}$. **(C)**

5.4. Use formula (5.18).

$$e_{50} = \sum_{t=1}^{\infty} e^{-t\mu} = 24$$

Summing up the geometric series with ratio $e^{-\mu}$:

$$\begin{aligned}
\frac{e^{-\mu}}{1-e^{-\mu}} &= 24 \\
e^{-\mu} &= \frac{24}{25} \\
\mu &= -\ln 0.96 = \boxed{\mathbf{0.040822}}
\end{aligned}$$

5.5. Under uniform distribution, life expectancy is the midrange. If life expectancy at 30 is 30, then ω must be $30 + 2(30) = 90$. Then $q_{30} = \frac{1}{\omega - x} = \boxed{\mathbf{\frac{1}{60}}}$. **(B)**

5.6. A repeat of the previous question with the age changed.

Under uniform distribution, life expectation is half of remaining life, so $\omega = 20 + 2(30) = 80$. Then $q_{20} = \frac{1}{80-20} = \boxed{\mathbf{\frac{1}{60}}}$. **(A)**

5.7. The current type is uniform, so $\omega = 2\mathring{e}_0 = 40$. The new type is sure to survive 5 years, then survival is uniform over 35 years thereafter, so the conditional expectation after surviving 5 years is $\frac{1}{2}(35) = 17.5$, for total expected survival time of $5 + 17.5 = 22.5$, which is $\boxed{2.5}$ years longer. **(B)**

5.8. This is a uniform distribution with $\omega = 72$ in months. Remaining time to ω (72) is 60, so life expectancy is 30. The number of eggs is then the number of hens times eggs per month times life expectancy, or $(100)(30)(30) = \boxed{90{,}000}$. **(E)**

5.9. Under uniform distribution, life expectancy is half the limiting age, so if expectancy increases 4, the limiting age increases by 8. $\boxed{108}$ **(E)**

5.10. Use equation (5.12). From the given information, we have two equations for $\omega - 20$ and n:

$$\frac{(\omega - 20 - 2n)(2n)}{\omega - 20} + \frac{4n^2}{2(\omega - 20)} = 25$$
$$\frac{(\omega - 20 - 4n)(4n)}{\omega - 20} + \frac{16n^2}{2(\omega - 20)} = 40$$

or

$$2n(\omega - 20) - 2n^2 = 25(\omega - 20)$$
$$4n(\omega - 20) - 8n^2 = 40(\omega - 20)$$

Solve for ω in both equations.

$$\omega - 20 = \frac{8n^2}{4n - 40} = \frac{2n^2}{2n - 25}$$

Solve this for n.

$$\begin{aligned} 2(4n - 40) &= 8(2n - 25) \\ -80 + 8n &= -200 + 16n \\ 8n &= 120 \\ n &= 15 \\ \omega - 20 &= \frac{2(15^2)}{2(15) - 25} = \frac{450}{5} = 90 \end{aligned}$$

Then

$$\mathring{e}_{20:\overline{45|}} = \frac{1}{2}(45) + \frac{1}{2}(22.5) = \boxed{33.75}$$

5.11. By equation (5.5), the difference between $\mathring{e}_{36:\overline{28|}}$ and $\int_0^{28} t\,{}_tp_{36}\,\mu_{36+t}\,\mathrm{d}t$ is $28\,{}_{28}p_{36}$. We calculate:

$$\begin{aligned} 28\,{}_{28}p_{36} &= 28\left(\frac{l_{64}}{l_{36}}\right) \\ &= 28\sqrt{\frac{36}{64}} = 21 \end{aligned}$$

Hence $\int_0^{28} t\,{}_tp_{36}\,\mu_{36+t}\,\mathrm{d}t = 24.67 - 21 = \boxed{3.67}$. **(A)**.

5.12. For the beta distribution, we know that $\mu_y = \frac{r}{\omega-y}$ and $\mathring{e}_y = (\omega-y)/(r+1)$. Therefore, using what we are given,

$$\frac{r}{\omega-y} = 0.1$$
$$\frac{\omega-y}{r+1} = 8.75$$

Multiplying these two equations together:

$$\frac{r}{r+1} = 0.875$$
$$r = \boxed{7} \qquad \textbf{(D)}$$

5.13. For the beta distribution, $\mu_x = \alpha/(\omega-x)$ and $\mathring{e}_x = (\omega-x)/(\alpha+1)$. Multiplying together gets **(A)**.

5.14. We are given $S_0(x) = \left(\dfrac{k^2-x}{k^2}\right)^{1/2}$. This is a beta with $\omega = k^2$ and $\alpha = 1/2$. Therefore,

$$\mathring{e}_x = \frac{\omega-x}{\alpha+1} = \frac{k^2-x}{3/2}$$

We are given that $\mathring{e}_{40} = 2\mathring{e}_{80}$, so

$$\frac{k^2-40}{3/2} = \frac{2(k^2-80)}{3/2}$$
$$k^2 = 120$$

Then $\mathring{e}_{60} = \dfrac{120-60}{3/2} = \boxed{40}$. **(D)**

5.15. We can write the denominator k as $(k^3)^{1/3}$, so

$$S_0(x) = \frac{(k^3-x)^{1/3}}{(k^3)^{1/3}} = \left(\frac{k^3-x}{k^3}\right)^{1/3}$$

This is a beta with $\omega = k^3$ and $\alpha = 1/3$. Therefore,

$$\mathring{e}_x = \frac{\omega-x}{\alpha+1} = \left(\frac{3}{4}\right)(k^3-x)$$

We are given that $\mathring{e}_{40} = 2\mathring{e}_{80}$, so

$$k^3 - 40 = 2(k^3-80)$$
$$k^3 = 120$$

Then $\mathring{e}_{60} = \left(\frac{3}{4}\right)(k^3-60) = \left(\frac{3}{4}\right)(60) = \boxed{45}$. **(B)**

5.16. The total number of patients hospitalized 21 days or longer is obtained by linear interpolation between 21 and 25:

$$l_{21} = 0.8(53{,}488) + 0.2(17{,}384) = 46{,}267.2$$

That will be the denominator. The numerator is the number of days past day 21 hospitalized times the number of patients hospitalized for that period. Within each interval of durations, the average patient released during that interval is hospitalized for half the period. So $46{,}267.2 - 17{,}384 = 28{,}883.2$ patients are hospitalized for 2 days after day 21, $17{,}384 - 5{,}349 = 12{,}035$ for $4 + 2.5 = 6.5$ days, $5{,}349 - 1{,}337 = 4{,}012$ for 11.5 days, and 1,337 for 16.5 days. Add it up:

$$28{,}883.2(2) + 12{,}035(6.5) + 4{,}012(11.5) + 1{,}337(16.5) = 204{,}192.4$$

The mean residual time is $204{,}192.4/46{,}267.2 = \boxed{4.41333}$. **(A)**

5.17. The question was asked when the Exam C distribution tables were part of this exam. The survival distribution is a two-parameter Pareto, and you could look up the mean in the tables. Then the question is easy: Complete life expectancy $\mathring{e}_x$ is listed in the tables as $(\theta + x)/(\alpha - 1)$ for a two-parameter Pareto. From this formula, it immediately follows that $\mathring{e}_{20} = (50 + 20)/(3 - 1) = \boxed{35}$.

To solve it from basic principles:

$$\begin{aligned}
{}_tp_{20} &= \frac{S_0(20+t)}{S_0(20)} \\
&= \frac{50^3/\big(50 + (20+t)\big)^3}{50^3/(50+20)^3} \\
&= \left(\frac{70}{70+t}\right)^3
\end{aligned}$$

From equation (5.2),

$$\begin{aligned}
\mathring{e}_{20} &= \int_0^\infty {}_tp_{20}\,\mathrm{d}t \\
&= \int_0^\infty \left(\frac{70}{70+t}\right)^3 \mathrm{d}t \\
&= \frac{70^3}{(2)(70^2)} = \boxed{35} \qquad \textbf{(D)}
\end{aligned}$$

5.18. The survival probability is

$${}_tp_0 = \frac{(200-t)(100-t)}{20{,}000} = \frac{(100-t)^2 + 100(100-t)}{20{,}000}$$

where the second equality, based on $200 - t = 100 - t + 100$, is used to make the coming integration easier. Then

$$\begin{aligned}
\mathring{e}_0 &= \frac{1}{20{,}000}\left(\int_0^{100} \big((100-t)^2 + 100(100-t)\big)\mathrm{d}t\right) \\
&= \frac{1}{20{,}000}\left(\frac{100^3}{3} + \frac{100^3}{2}\right) = \frac{833{,}333\frac{1}{3}}{20{,}000} = \boxed{41\tfrac{2}{3}}
\end{aligned}$$

5.19. Mortality is uniformly distributed. By equation (5.24),

$$e_{10:\overline{20}|} = \mathring{e}_{10:\overline{20}|} - 0.5\,{}_{20}q_{10}$$

However, by equation (5.12),

$$\mathring{e}_{10:\overline{20}|} = 20\,{}_{20}p_{10} + 10\,{}_{20}q_{10}$$

Putting these two equations together,

$$e_{10:\overline{20}|} = 20\,{}_{20}p_{10} + 9.5\,{}_{20}q_{10}$$

We are given that $e_{10:\overline{20}|} = 18$, so

$$\begin{aligned} 18 &= 20\,{}_{20}p_{10} + 9.5\,{}_{20}q_{10} \\ &= 20 - 10.5\,{}_{20}q_{10} \\ {}_{20}q_{10} &= \frac{2}{10.5} = \frac{4}{21} \end{aligned}$$

For uniform mortality, ${}_tq_x = t/(\omega - x)$, so ${}_{20}q_{10} = 20/(\omega - 10)$.

$$\begin{aligned} \frac{20}{\omega - 10} &= \frac{4}{21} \\ \omega &= \boxed{\mathbf{115}} \end{aligned}$$

5.20. Survival time is uniformly distributed with $\omega = 100$. Therefore

$$\begin{aligned} \mathring{e}_{90} &= \frac{10}{2} = 5 \\ e_{90} &= \mathring{e}_{90} - 0.5 = \boxed{\mathbf{4.5}} \qquad \textbf{(C)} \end{aligned}$$

5.21. Calculating ${}_tp_{40} = {}_{10}p_{40}\,{}_{t-10}p_{50}$ for $t > 10$, we get

$${}_tp_{40} = \begin{cases} e^{-0.01t} & t \le 10 \\ e^{-0.01(10)}e^{-0.01(t-10)}\left(\frac{50-(t-10)}{50}\right) = e^{-0.01t}\left(\frac{60-t}{50}\right) & t \ge 10 \end{cases}$$

We integrate ${}_tp_{40}$ to calculate $\mathring{e}_{40}$.

$$\mathring{e}_{40} = \int_0^{10} e^{-0.01t}\,\mathrm{d}t + \int_{10}^{60} e^{-0.01t}\left(\frac{60-t}{50}\right)\mathrm{d}t$$

The first integral is $100(1 - e^{-0.1})$. The second integral is integrated by parts. We integrate $\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$, where $u = \left(\frac{60-t}{50}\right)$ and $\mathrm{d}v = e^{-0.01t}\,\mathrm{d}t$.

$$\begin{aligned} \int_{10}^{60} e^{-0.01t}\left(\frac{60-t}{50}\right)\mathrm{d}t &= \left(\frac{e^{-0.01t}}{-0.01}\right)\left(\frac{60-t}{50}\right)\Bigg|_{10}^{60} - \int_{10}^{60}\left(\frac{e^{-0.01t}}{0.01}\right)\left(\frac{1}{50}\right)\mathrm{d}t \\ &= 100e^{-0.1} - \frac{e^{-0.01t}}{-0.01^2(50)}\Bigg|_{10}^{60} \\ &= 100e^{-0.1} - 200(e^{-0.1} - e^{-0.6}) \end{aligned}$$

$$= -100e^{-0.1} + 200e^{-0.6}$$

Putting everything together,

$$\begin{aligned}\mathring{e}_{40} &= 100(1-e^{-0.1}) - 100e^{-0.1} + 200e^{-0.6}\\ &= 100 - 200e^{-0.1} + 200e^{-0.6} = \boxed{\mathbf{28.7948}}\end{aligned}$$

5.22. We use equation (5.6):

$$\mathbf{E}[\min(X,5000)] = \int_0^{5000} S_0(x)\mathrm{d}x = \int_0^{4800} 1\,\mathrm{d}x + \int_{4800}^{5000}\left(1-\frac{t-4800}{1200}\right)\mathrm{d}x$$

The first integral is 4800. The second integral is the integral of a trapezoid with heights 1 at 4800 and 5/6 at 5000, and width 200. The area of the trapezoid is $200\left(\frac{1}{2}\right)\left(\frac{11}{6}\right) = 183\frac{1}{3}$. The complete temporary life expectancy to 5000 is $\mathbf{E}[\min(X,5000)] = 4800 + 183\frac{1}{3} = 4983\frac{1}{3}$. The number of kilowatt-hours used by 50 bulbs is $50(0.015)(4983\frac{1}{3}) = \boxed{\mathbf{3737.5}}$.

Here is a more intuitive way to solve the question. Survival for 4800 hours is definite. After 4800 hours, survival is uniform for 1200 hours. Therefore, 5/6 of the bulbs survive to time 5000 (1/6 of the interval [4800,6000]), and the remaining 1/6 of the bulbs survive for half of the interval [4800,5000], or to time 4900 on the average. Therefore, average survival for all bulbs is $5/6(5000) + 1/6(4900) = 4983\frac{1}{3}$, and the number of kilowatt-hours used by 50 bulbs is once again $50(0.015)(4983\frac{1}{3}) = \boxed{\mathbf{3737.5}}$.

5.23. Since $\mathring{e}_{20} = 45$, $\omega = 20 + 2(45) = 110$, and future lifetime is uniform over $[0,90]$. Then the variance is the length of the interval squared over 12, or $90^2/12 = \boxed{\mathbf{675}}$. **(D)**

5.24. Variance is $(\omega-50)^2/12 = 192$, so $\omega - 50 = 48$, $\omega = \boxed{\mathbf{98}}$. **(A)**

5.25. Future lifetime is uniformly distributed on $[0,80]$, so the variance is $80^2/12 = \boxed{\mathbf{533\frac{1}{3}}}$. **(A)**

5.26. Let $\theta = \omega - 30$.

The straightforward approach is to calculate the first and second moments of the minimum of 10 and future lifetime for (30). Since mortality is uniform, this may be calculated by conditioning on survival to time 40. The first moment is

$$\begin{aligned}\mathring{e}_{30:\overline{10}|} &= {}_{10}p_{30}(10) + {}_{10}q_{30}\,\mathbf{E}[T_{30} \mid T_{30} \le 10]\\ &= \left(\frac{\theta-10}{\theta}\right)(10) + \left(\frac{10}{\theta}\right)(5) = \frac{10\theta-50}{\theta}\end{aligned}$$

To calculate the second moment, for a uniform distribution on $[0,n]$, the second moment is $n^2/3$, so given that death occurs within 10 years, the second moment of survival time is 100/3.

$$\mathbf{E}[\min(T_{30},10)^2] = \left(\frac{\theta-10}{\theta}\right)(10^2) + \left(\frac{10}{\theta}\right)\left(\frac{100}{3}\right) = \frac{300\theta-2000}{3\theta}$$

Alternatively, the first and second moments can be calculated using integration. The first moment is

$$\begin{aligned}\mathbf{E}[\min(T_{30},10)] &= \int_0^{10} {}_tp_{30}\,\mathrm{d}t\\ &= \int_0^{10}\frac{(\theta-t)\mathrm{d}t}{\theta} = 10 - \frac{50}{\theta}\end{aligned}$$

and the second moment, using formula (5.7) on page 80 is

$$\begin{aligned}
\mathbf{E}\left[\left(\min(T_{30},10)\right)^2\right] &= 2\int_0^{10} t\,{}_tp_{30}\,\mathrm{d}t \\
&= 2\int_0^{10} \frac{t(\theta-t)\mathrm{d}t}{\theta} \\
&= \frac{2}{\theta}\left(50\theta - \frac{1000}{3}\right) \\
&= 100 - \frac{2000}{3\theta}
\end{aligned}$$

Either way, the variance is then

$$\begin{aligned}
\operatorname{Var}\bigl(\min(T_{30},10)\bigr) &= 100 - \frac{2000}{3\theta} - 100 + \frac{1000}{\theta} - \frac{2500}{\theta^2} \\
&= -\frac{2500}{\theta^2} + \frac{1000}{3\theta}
\end{aligned}$$

This is set equal to 3.5755 and the quadratic is solved.

An alternative way to get the same quadratic which may be easier is to use the conditional variance formula (1.13) on page 9. We condition on surviving 10 years. Given that (30) survives 10 years, the amount of time (30) survives in the next 10 years is exactly 10 years, so the expected value of the amount of time survived in the next 10 years is 10 and the variance is 0. If (30) does not survive 10 years, death time is uniformly distributed on $[0,10]$. The expected value of a uniform distribution is the midrange or 5, and the variance is the range squared divided by 12, or $\frac{10^2}{12} = \frac{25}{3}$. The probability of dying within 10 years is $\frac{10}{\theta}$. Therefore:

- The expected value of the two variances is $\left(\frac{10}{\theta}\right)\left(\frac{25}{3}\right)$.
- The variance of the expected values (using the Bernoulli shortcut, Section 1.2.1 on page 3) is $\left(\frac{10}{\theta}\right)\left(\frac{\theta-10}{\theta}\right)(25)$.

So the variance of 10-year future lifetime for (30) is

$$\begin{aligned}
\left(\frac{10}{\theta}\right)\left(\frac{\theta-10}{\theta}\right)(25) + \left(\frac{10}{\theta}\right)\left(\frac{25}{3}\right) &= 3.5755 \\
\frac{250}{\theta} - \frac{2500}{\theta^2} + \frac{250}{3\theta} &= 3.5755 \\
\frac{2500}{\theta^2} - \frac{1000}{3\theta} + 3.5755 &= 0
\end{aligned}$$

Either way yields this quadratic. Using the quadratic formula for $\frac{1}{\theta}$, we get the solutions

$$\begin{aligned}
\frac{1}{\theta} &= \frac{-\frac{1000}{3} \pm \sqrt{\left(\frac{1000}{3}\right)^2 - 35{,}755}}{2(2500)} \\
\frac{1}{\theta} &= 0.011765, 0.121569 \\
\theta &= 85.00, 8.22579
\end{aligned}$$

Since we are given that $\omega > 40$, the answer is $30 + 85.00 =$ **115.00**.

Note that if ω were less than 40, then the variance in number of years lived between 30 and 40 is variance of complete future lifetime. The variance of a uniform distribution on $[0,\theta]$ is $\frac{\theta^2}{12}$. Since future lifetime is uniformly distributed, the variance is the maximum number of future years squared over 12. So we'd have

$$\begin{aligned}\frac{\theta^2}{12} &= 3.5755\\ \theta &= 6.5503\end{aligned}$$

and ω would then be 36.5503.

5.27. This came from a written answer question.

For (i), because of the lack of memory of an exponential distribution, we can evaluate it as ${}_{5|10}q_0$:

$$\frac{S_0(5)-S_0(15)}{S_0(0)} = e^{-0.25} - e^{-0.75} = \boxed{\mathbf{0.306434}}$$

(ii) is $1 - S_0(30) = 1 - e^{-1.5} = \boxed{\mathbf{0.776870}}$.

(iii) is the reciprocal of the force of mortality, or $\boxed{\mathbf{20}}$.

For (iv), variance is the square of the mean for an exponential, or $\boxed{\mathbf{400}}$.

5.28. The variance of a uniform distribution is the square of time to ω divided by 12.

$$\begin{aligned}\frac{(\omega-15)^2}{12} &= 675\\ (\omega-15)^2 &= 8100\\ \omega &= 105\end{aligned}$$

The complete expectation of life is half the maximum remaining life ($\omega - 25 = 80$), or $\boxed{\mathbf{40}}$. **(B)**

5.29. Age at death is uniformly distributed with $\omega = 2\mathring{e}_0 = 50$. Then

$$\operatorname{Var}(T_{10}) = \frac{(50-10)^2}{12} = \boxed{\mathbf{133\tfrac{1}{3}}} \qquad \textbf{(C)}$$

5.30. This is variance of curtate lifetime. We'll use formula (5.19). Expected value is

$$\begin{aligned}e_{67:\overline{3}|} &= p_{67} + {}_2p_{67} + {}_3p_{67}\\ &= \frac{l_{68}+l_{69}+l_{70}}{l_{67}}\\ &= \frac{7{,}018{,}432 + 6{,}823{,}367 + 6{,}616{,}155}{7{,}201{,}635} = 2.840737\end{aligned}$$

The second moment is

$$\begin{aligned}\mathbf{E}\Big[\big(\min(K_{67},3)\big)^2\Big] &= p_{67} + 3\,{}_2p_{67} + 5\,{}_3p_{67}\\ &= \frac{l_{68}+3l_{69}+5l_{70}}{l_{67}}\\ &= \frac{7{,}018{,}432 + 3(6{,}823{,}367) + 5(6{,}616{,}155)}{7{,}201{,}635} = 8.410493\end{aligned}$$

The variance is $8.410493 - 2.840737^2 = \boxed{\mathbf{0.340706}}$

5.31. The survival probabilities are

$$\begin{aligned} p_x &= 0.7 \\ {}_2p_x &= (0.7)(0.5) = 0.35 \\ {}_3p_x &= (0.35)(0.3) = 0.105 \\ {}_4p_x &= 0 \end{aligned}$$

The moments are

$$\begin{aligned} \mathbf{E}[K_x] &= 0.7 + 0.35 + 0.105 = 1.155 \\ \mathbf{E}[K_x^2] &= 2\big(0.7 + 0.35(2) + 0.105(3)\big) - 1.155 = 2.275 \\ \operatorname{Var}(K_x) &= 2.275 - 1.155^2 = \boxed{\mathbf{0.940975}} \end{aligned}$$

5.32. For constant force of mortality, the curtate life expectancy is $e^{-\mu}/(1-e^{-\mu})$ by equation (5.22). Setting $e_{35} = 49$ and solving for μ, we get

$$\begin{aligned} 49 &= \frac{e^{-\mu}}{1-e^{-\mu}} \\ 49 - 49e^{-\mu} &= e^{-\mu} \\ e^{-\mu} &= \frac{49}{50} \\ \mu &= 0.020203 \end{aligned}$$

Using primes for revised functions, adding 0.01, the revised μ' is 0.030203. Then

$$e'_{35} = \frac{e^{-0.030203}}{1-e^{-0.030203}} = \boxed{\mathbf{32.61213}}$$

5.33. This could be done from the definition, and such a solution is provided in the official solutions. But we will use formulas (5.19) and (5.21). The survival probabilities are

$$\begin{aligned} p_x &= 0.9 \\ {}_2p_x &= (0.9)(0.8) = 0.72 \\ {}_3p_x &= (0.9)(0.8)(0.7) = 0.504 \end{aligned}$$

The moments are

$$\begin{aligned} \mathbf{E}[X] &= 0.9 + 0.72 + 0.504 = 2.124 \\ \mathbf{E}\left[X^2\right] &= 0.9(1) + 0.72(3) + 0.504(5) = 5.58 \\ \operatorname{Var}(X) &= 5.58 - 2.124^2 = \boxed{\mathbf{1.068624}} \qquad \textbf{(A)} \end{aligned}$$

5.34. Survival is uniform. Complete temporary life expectancy to age 70 is therefore equal to the probability of death before age 70 times the median time to death (20), plus 40 times the probability of death after age 70. Setting this equal to 27.692,

$$\begin{aligned} 20\,{}_{40}q_{30} + 40(1 - {}_{40}q_{30}) &= 27.692 \\ 40 - 20\,{}_{40}q_{30} &= 27.692 \\ {}_{40}q_{30} &= \frac{40 - 27.692}{20} = 0.6154 \end{aligned}$$

But ${}_{40}q_{30} = 40/(\omega - 30)$, so $\omega - 30 = 40/0.6154 = 65$. The variance of a uniform distribution is the interval squared over 12. Here the interval for (30) is $\omega - 30 = 65$, so the answer is $65^2/12 = \boxed{\mathbf{352\tfrac{1}{12}}}$. **(B)**

5.35. Use formula (5.18). Mortality follows a beta distribution with $\alpha = 2$ (multiply numerator and denominator of μ_x by 2 to see this). Therefore

$$
\begin{aligned}
{}_kp_{30} &= \left(\frac{80-k}{80}\right)^2 \\
e_{30} &= \sum_{k=1}^{79}\left(\frac{80-k}{80}\right)^2
\end{aligned}
$$

Notice that the numerators constitute a sum of squares from 1 to 79.

$$e_{30} = \frac{1}{80^2}\sum_{k=1}^{79} k^2 = \frac{(79)(80)(159)}{(80^2)(6)} = \boxed{26.16875} \qquad \textbf{(A)}$$

You would get the correct answer choice using the approximation $e_{30} \approx \mathring{e}_{30} - 0.5$.

5.36. Mortality follows a beta distribution. In the paragraph before example 5B on page 82 we mention that if X follows a beta distribution, then

$$
\begin{aligned}
\mathbf{E}[X] = \mathring{e}_0 &= \frac{\omega}{\alpha+1} \\
\mu_x &= \frac{\alpha}{\omega - x}
\end{aligned}
$$

Let α' be the revised α. We are given

$$
\begin{aligned}
\frac{\omega}{\alpha'+1} &= \frac{\omega}{2(\alpha+1)} && \text{from (i)} \\
\frac{\alpha'}{\omega-x} &= \frac{2.25\alpha}{\omega-x} && \text{from (ii)}
\end{aligned}
$$

From the second equation, $\alpha' = 2.25\alpha$. From the first equation,

$$
\begin{aligned}
2.25\alpha + 1 &= 2\alpha + 2 \\
0.25\alpha &= 1 \\
\alpha &= \boxed{4} \qquad \textbf{(D)}
\end{aligned}
$$

5.37. Mortality is exponential with variance 100 and therefore mean $\sqrt{100} = 10$; ${}_tp_x = e^{-x/10}$. To calculate $\mathbf{E}[\min(T,10)]$, the 10-year temporary complete life expectancy, we use formula (5.6).

$$\int_0^{10} e^{-x/10}\,\mathrm{d}x = 10\left(1 - e^{-1}\right) = \boxed{6.3212} \qquad \textbf{(C)}$$

Alternatively, $\mathbf{E}[X]$ is the complete expectation now minus the probability of survival for 10 years times the complete expectation 10 years from now, or

$$E[X] = \mathring{e}_{x:\overline{10}|} = \mathring{e}_x - {}_{10}p_x\,\mathring{e}_{x+10} = 10(1 - e^{-10(0.1)}) = 6.3212$$

5.38. We use equation (5.6) in conjunction with ${}_tp_{30} = S_0(30+t)/S_0(30)$.

$$\mathring{e}_{30:\overline{50|}} = \int_0^{50} {}_tp_{30}\,\mathrm{d}t = \frac{\int_{30}^{80}\left(1-(0.01t)^2\right)\mathrm{d}t}{S_0(30)}$$

$$S_0(30) = 1-\big(0.01(30)\big)^2 = 0.91$$

$$\int_{30}^{80}\left(1-(0.01t)^2\right)\mathrm{d}t = 50 - \frac{0.01^2(80^3-30^3)}{3} = 33\tfrac{5}{6}$$

$$\mathring{e}_{30:\overline{50|}} = \frac{33\frac{5}{6}}{0.91} = \boxed{37.179} \qquad \textbf{(D)}$$

5.39. Since ${}_2p_{70}$ is equal for all three models and $e_{70:\overline{2|}} = p_{70} + {}_2p_{70}$, the answer depends purely on p_{70}. The curve ${}_tp_{70}$ for $t \in [0,2]$ is a line for DM and a convex curve lying below the line for CF (since they intersect at $t=0$ and $t=2$), so CF < DM. For ILT, we find that $d_{70} = 6{,}615{,}155 - 6{,}396{,}609 = 218{,}546$ while $d_{71} = 6{,}396{,}609 - 6{,}164{,}663 = 231{,}946$. The number of deaths under DM is the same in both years, so it must be less than ILT in the first year, making $p_{70}^{\text{ILT}} > p_{70}^{\text{DM}}$ and DM > CF. Thus **(C)** is the correct choice.

5.40. The new model is beta, so the expected lifetime is $\mathring{e}_x = (\omega - x)/(\alpha+1)$ and can set up simultaneous equations to solve for ω:

$$\frac{\omega-30}{\alpha+1} = \frac{4}{3}\left(\frac{\omega-30}{2}\right) \qquad \text{from (ii)}$$

$$\frac{\omega-60}{\alpha+1} = 20 \qquad \text{from (iii)}$$

From the second equation, $\alpha + 1 = (\omega-60)/20$ which we plug into the first equation.

$$\frac{20(\omega-30)}{\omega-60} = \frac{4}{3}\left(\frac{\omega-30}{2}\right)$$

$$\frac{20}{\omega-60} = \frac{4}{6} = \frac{2}{3}$$

$$\omega - 60 = \frac{20}{2/3} = 30$$

$$\omega = 90$$

Then in the previous uniform model, $\mathring{e}_{70} = (90-70)/2 = \boxed{10}$. **(B)** It is not necessary to solve for α, but $\alpha = 0.5$.

5.41. The mortality rates for the 2011 person are

$$q_{71} = 0.012 \qquad p_{71} = 0.988$$

$$q_{72} = 0.015(0.984) = 0.01476 \qquad {}_2p_{71} = (0.988)(1-0.01476) = 0.973417$$

$$q_{73} = 0.02(0.986)^2 = 0.0194439 \qquad {}_3p_{71} = 0.973417(1-0.0194439) = 0.954490$$

The 3-year temporary curtate life expectancy is $0.988 + 0.973417 + 0.954490 = 2.9159$.

For the 2016 person, multiply the above mortality rates by r^5 where r is the reduction factor for that age.

$$q_{71} = 0.012(0.982)^5 = 0.010958 \qquad p_{71} = 0.989042$$

$$q_{72} = 0.01476(0.984)^5 = 0.013616 \qquad {}_2p_{71} = (0.989042)(1 - 0.013616) = 0.975575$$
$$q_{73} = 0.0199439(0.986)^5 = 0.018120 \qquad {}_3p_{71} = (0.975575)(1 - 0.018120) = 0.957897$$

The 3-year temporary curtate life expectancy is $0.989042 + 0.975575 + 0.957897 = 2.9225$

The difference is **0.0066**.

Quiz Solutions

5-1. Calculate ${}_tp_{50}$.

$$\begin{aligned}\int_{50}^{50+t} \mu_x \,\mathrm{d}x &= -\ln(10 - \sqrt{x})\Big|_{50}^{50+t} \\ &= \ln\left(10 - \sqrt{50}\right) - \ln\left(10 - \sqrt{50+t}\right) \\ {}_tp_{50} = \exp\left(-\int_{50}^{50+t} \mu_x \,\mathrm{d}x\right) &= \frac{10 - \sqrt{50+t}}{10 - \sqrt{50}}\end{aligned}$$

Calculate $\mathring{e}_{50}$ using formula (5.2).

$$\begin{aligned}\mathring{e}_{50} &= \int_0^{50} \frac{10 - \sqrt{50+t}}{10 - \sqrt{50}} \mathrm{d}t \\ &= \frac{1}{10 - \sqrt{50}} \left(500 - \frac{2}{3}(50+t)^{3/2}\Big|_0^{50}\right) \\ &= \frac{1}{10 - \sqrt{50}} \left(500 - \frac{2}{3}(100^{3/2} - 50^{3/2})\right) \\ &= \frac{69.03559}{2.92893} = \boxed{23.57}\end{aligned}$$

5-2. Set up the equation for $\mathring{e}_{20:\overline{n}|}$.

$$\begin{aligned}\mathring{e}_{20:\overline{n}|} &= \frac{100-n}{100}(n) + \frac{n}{100}\left(\frac{n}{2}\right) \\ 4800 &= 100n - n^2 + 0.5n^2 = -0.5n^2 + 100n\end{aligned}$$

We therefore have the quadratic $n^2 - 200n + 9600 = 0$. Solutions are $n = \boxed{80}$ and $n = 120$. However, 120 is spurious since $20 + n$ must be less than ω.

5-3. First calculate the mean:

$$e_{90} = 0.98 + 0.95 = 1.93$$

Then the second moment, using formula (5.20):

$$\mathbf{E}\left[K_{90}^2\right] = 0.98 + 0.95(3) = 3.83$$

The variance is $\mathrm{Var}(K_{90}) = 3.83 - 1.93^2 = \boxed{0.1051}$.

5-4. Using $\mathring{e}_{20:\overline{24}|} = 24\,{}_{24}p_{20} + 12\,{}_{24}q_{20}$ and $e_{20:\overline{24}|} = \mathring{e}_{20:\overline{24}|} - 0.5\,{}_{24}q_{20}$, we have

$$\begin{aligned}
20.25 &= 24\,{}_{24}p_{20} + 11.5\,{}_{24}q_{20}\\
&= 24 - 24\,{}_{24}q_{20} + 11.5\,{}_{24}q_{20}\\
{}_{24}q_{20} &= \frac{3.75}{12.5} = 0.3\\
\frac{24}{\omega - 20} &= 0.3\\
\omega - 20 &= 80\\
\omega &= \boxed{\mathbf{100}}
\end{aligned}$$

Lesson 6

Survival Distributions: Percentiles and Recursions

Reading: Not directly discussed in *Actuarial Mathematics for Life Contingent Risks* 2nd edition

Importance of this lesson: The material in this lesson is important, even though it is not directly discussed in *Actuarial Mathematics for Life Contingent Risks*. The exam will ask questions on percentiles/probabilities of insurance, annuities, and losses (topics discussed later on in the course), and all of those questions depend on the material here.

6.1 Percentiles

A 100π percentile of survival time is the time t such that there is a $100\pi\%$ probability that survival time is less than t. In other words, it is t such that ${}_tq_x = \pi$, or ${}_tp_x = 1 - \pi$. Notice that high percentiles correspond to low probabilities of survival.

A special case is the median remaining lifetime at age x, which is t such that ${}_tp_x = {}_tq_x = 0.50$.

If future survival time is uniformly distributed with limiting age ω, median remaining lifetime at age x is the midrange, half way from x to ω, or $(\omega - x)/2$. If survival has a constant force of mortality μ, meaning that it follows an exponential distribution, we must find t such that $e^{-\mu t} = 0.5$. Solving this, we have that $t = (\ln 2)/\mu$. It doesn't matter what x is in this case since the exponential distribution is memoryless. Here's an example that's a little harder:

Example 6A A person age 70 is subject to the following force of mortality:

$$\mu_{70+t} = \begin{cases} 0.1 & t \le 5 \\ 0.2 & t > 5 \end{cases}$$

Calculate median future lifetime for this person.

Answer: We want ${}_tp_x = 0.5$. For $t \le 5$, ${}_tp_x = e^{-0.1t}$. Plugging in $t = 5$, we get $e^{-0.5} = 0.606531$, which is greater than 0.5, so the median is greater than 5. For $t > 5$, ${}_tp_x = e^{-0.5}e^{-0.2(t-5)}$. We solve:

$$\begin{aligned} e^{-0.5}e^{-0.2(t-5)} &= 0.5 \\ -0.5 - 0.2t + 1 &= \ln 0.5 \\ t &= \frac{-\ln 0.5 + 0.5}{0.2} = \boxed{5.9657} \end{aligned}$$ □

If you are using a life table, then the 100π percentile of future lifetime at age x is the age $x + t$ at which l_{x+t} is equal to $(1 - \pi)l_x$. For example, if $l_x = 1{,}000{,}000$, then the 20th percentile is the t such that $l_{x+t} = 800{,}000$. Usually there will be no t satisfying this exactly, so you will only know the median is between two integral ages. To get an exact answer, it will be necessary to make some assumption about mortality between the two ages and then to interpolate between the ages. Interpolation is discussed in the next lesson.

Example 6B Mortality follows the Illustrative Life Table.

Determine the age containing the 90th percentile of survival time for a person age (30).

Answer: $l_{30} = 9{,}501{,}381$, so we need the age x such that $l_x = 0.1 l_{30} = 950{,}138.1$. We see that $l_{90} = 1{,}058{,}491$ is higher and $l_{91} = 858{,}676$ is lower, so age **90** is the age containing the 90th percentile. □

Quiz 6-1 Future lifetime is subject to force of mortality

$$\mu_{x+t} = \frac{1}{t+50}$$

Determine the third quartile of future lifetime.

6.2 Recursive formulas for life expectancy

For both complete and curtate future lifetime, we have formulas expressing them as the sums or integrals of probabilities of survival. We can break the formula for the life expectancy of (x) up into a component for temporary life expectancy for a period of n years, plus a component involving the life expectancy of $(x+n)$ times the probability of surviving n years. A special case is when $n = 1$.

For complete life expectancy, the decomposition looks like this:

$$\mathring{e}_x = \mathring{e}_{x:\overline{n}|} + {}_n p_x\, \mathring{e}_{x+n} \tag{6.1}$$

Think about the meaning of this equation. It says that the average number of years of future life equals the average number of years of future life over the next n years, plus the probability of survival for n years times the average number of years of future life past n years. You can easily derive the equation algebraically, but understanding what it means will help you reproduce it.

For curtate life expectancy, the decomposition looks like this:

$$e_x = e_{x:\overline{n}|} + {}_n p_x\, e_{x+n} \tag{6.2}$$

$$= e_{x:\overline{n-1}|} + {}_n p_x (1 + e_{x+n}) \tag{6.3}$$

and when $n = 1$

$$e_x = p_x + p_x\, e_{x+1} = p_x(1 + e_{x+1}) \tag{6.4}$$

Equation (6.4) is the most commonly used equation of the four presented here, and does not require calculation of any temporary life expectancies. It allows recursive construction of a table of life expectancies at all ages. To construct such a table, start with the end of the table ω, at which $e_\omega = 0$, and then calculate e_{x-1} from e_x repeatedly starting with $x = \omega$ and ending with $x = 1$.

The recursive formulas allow fast computation of life expectancy when you are given a piecewise constant force of mortality, as the next example shows.

Example 6C A person age 70 is subject to the following force of mortality:

$$\mu_{70+t} = \begin{cases} 0.01 & t \le 5 \\ 0.02 & t > 5 \end{cases}$$

Calculate $\mathring{e}_{70}$ for this person.

Answer: This is the same as exercise 5.1, but we will now do it without integrals.

By the recursive formula,

$$\mathring{e}_{70} = \mathring{e}_{70:\overline{5}|} + {}_5p_{70}\,\mathring{e}_{75}$$

Now, $\mathring{e}_{75}$ is the life expectancy of someone with constant force of mortality 0.02, or exponential mortality, and we know that the life expectancy for exponential mortality is the reciprocal of the force, or $\mathring{e}_{75} = 1/0.02 = 50$. Also, ${}_5p_{70} = e^{-0.01(5)} = e^{-0.05}$.

Now, consider a person age 70 subject to the constant force of mortality 0.01. We will use primes for this person's mortality functions. By the recursive formula,

$$\mathring{e}'_{70} = \mathring{e}'_{70:\overline{5}|} + {}_5p'_{70}\,\mathring{e}'_{75}$$

However, $\mathring{e}'_{70} = \mathring{e}'_{75} = 1/0.01 = 100$ and ${}_5p'_{70} = e^{-0.05}$, so $\mathring{e}'_{70:\overline{5}|} = 100\left(1 - e^{-0.05}\right)$. And now the punch line: $\mathring{e}'_{70:\overline{5}|} = \mathring{e}_{70:\overline{5}|}$. Why? Because for the first five years, the forces of mortality for the person in our example and the person with constant force 0.01 are the same, and $\mathring{e}_{70:\overline{5}|}$ is a function only of the force of mortality in the first five years. So we have

$$\mathring{e}_{70} = 100\left(1 - e^{-0.05}\right) + 50e^{-0.05} = 4.8771 + 47.5615 = \boxed{\mathbf{52.4386}}$$ □

A common use of recursive computation on exam questions is calculating the effect on life expectancy of changing the mortality assumption for a year or for a period. You would start with the original e_x, then advance to the age $x + n$ that is beyond the change in mortality, then work back to e_x using the revised mortality assumptions.

Example 6D For (x), standard curtate life expectancy is 72 years and standard $q_x = 0.01$. Because (x) has better underwriting characteristics, $q_x = 0.005$ for (x), but mortality for ages $x + 1$ and higher is standard.

Calculate curtate life expectancy for (x).

Answer: We calculate e_{x+1} by rearranging equation (6.4):

$$\begin{aligned} e_{x+1} &= \frac{e_x - p_x}{p_x} \\ &= \frac{72 - 0.99}{0.99} = 71.72727 \end{aligned}$$

Then for (x), we have

$$e_x = 0.995 + 0.995(71.72727) = \boxed{\mathbf{72.36364}}$$ □

The recursive formulas also work for temporary life expectancies on the left side of the equation. For example, equations (6.2) and (6.3) become

$$e_{x:\overline{n}|} = e_{x:\overline{m}|} + {}_mp_x\, e_{x+m:\overline{n-m}|} \qquad m < n \tag{6.5}$$

$$\phantom{e_{x:\overline{n}|}} = e_{x:\overline{m-1}|} + {}_mp_x(1 + e_{x+m:\overline{n-m}|}) \qquad m < n \tag{6.6}$$

and equation (6.4) becomes

$$e_{x:\overline{n}|} = p_x + p_x\, e_{x+1:\overline{n-1}|} = p_x(1 + e_{x+1:\overline{n-1}|}) \tag{6.7}$$

Quiz 6-2 You are given:

(i) $e_0 = 70$

(ii) $e_1 > e_0$

Determine the least upper bound for p_0.

The formulas in this lesson are summarized in Table 6.1.

Table 6.1: Formula summary for this lesson

Recursive formulas

$$\mathring{e}_x = \mathring{e}_{x:\overline{n}|} + {}_np_x\,\mathring{e}_{x+n} \tag{6.1}$$

$$\mathring{e}_{x:\overline{n}|} = \mathring{e}_{x:\overline{m}|} + {}_mp_x\,\mathring{e}_{x+m:\overline{n-m}|}, \quad m < n$$

$$e_x = e_{x:\overline{n}|} + {}_np_x\,e_{x+n} \tag{6.2}$$

$$= e_{x:\overline{n-1}|} + {}_np_x(1 + e_{x+n}) \tag{6.3}$$

$$e_x = p_x + p_x\,e_{x+1} = p_x(1 + e_{x+1}) \tag{6.4}$$

$$e_{x:\overline{n}|} = e_{x:\overline{m}|} + {}_mp_x\,e_{x+m:\overline{n-m}|}, \quad m < n \tag{6.5}$$

$$= e_{x:\overline{m-1}|} + {}_mp_x(1 + e_{x+m:\overline{n-m}|}) \quad m < n \tag{6.6}$$

$$e_{x:\overline{n}|} = p_x + p_x\,e_{x+1:\overline{n-1}|} = p_x(1 + e_{x+1:\overline{n-1}|}) \tag{6.7}$$

Exercises

Percentiles

6.1. Mortality follows Gompertz's law, with $\mu_x = Bc^x$. You are given that the 10th percentile of future lifetime at birth is 40 and the 70th percentile is 80.

Determine c.

6.2. Mortality follows

$$S_0(x) = \left(\frac{120 - x}{120}\right)^{1/2} \qquad 0 \le x \le 120$$

Calculate the median future lifetime for (30).

6.3. **[4-F86:14]** You are given $S_0(x) = 1/(1 + x)$.

Determine the median future lifetime of (y).

(A) $y + 1$ (B) y (C) 1 (D) $\frac{1}{y}$ (E) $\frac{1}{1+y}$

6.4. **[150-S87:7; 150-S90:7** is virtually identical] Which of the following are true?

I. ${}_{t+u}q_x \ge {}_uq_{x+t}$ for $t \ge 0$ and $u \ge 0$.

II. ${}_uq_{x+t} \ge {}_{t|u}q_x$ for $t \ge 0$ and $u \ge 0$.

III. If $S_0(x)$ follows a uniform distribution, the median future lifetime of (x) equals the mean future lifetime of (x).

(A) I and II only (B) I and III only (C) II and III only (D) I, II and III
(E) The correct answer is not given by (A) , (B) , (C) , or (D) .

Exercises continue on the next page . . .

6.5. **[CAS4A-F93:20]** (2 points) You are given a survival function $S_0(x) = 1 - 0.01x$ for $0 < x \leq 100$.

Determine the median future lifetime of a life age 10.

(A) Less than 42
(B) At least 42, but less than 44
(C) At least 44, but less than 46
(D) At least 46, but less than 48
(E) At least 48

6.6. **[150-F96:1]** You are given $\mu_x = \sqrt{\frac{1}{80-x}}$, $0 \leq x < 80$.

Calculate the median future lifetime of (20).

(A) 5.25 (B) 6.08 (C) 8.52 (D) 26.08 (E) 30.00

6.7. **[SOA3-F03:18]** A population has 30% who are smokers with a constant force of mortality 0.2 and 70% who are non-smokers with a constant force of mortality 0.1.

Calculate the 75$^{\text{th}}$ percentile of the distribution of the future lifetime of an individual selected at random from this population.

(A) 10.7 (B) 11.0 (C) 11.2 (D) 11.6 (E) 11.8

6.8. A life age 45 is subject to Gompertz's law with $B = 0.0003$, and $c = 1.065$.

Determine the 60$^{\text{th}}$ percentile of future lifetime for this life.

(A) 24 (B) 28 (C) 32 (D) 36 (E) 40

Recursive Formulas

6.9. You are given the following mortality rates:

x	q_x
65	0.010
66	0.015
67	0.020
68	0.030
69	0.035

You are also given that $e_{65} = 24$.

Calculate e_{68}.

6.10. **[SOA3-F03:35]** For T, the future lifetime random variable for (0):

(i) $\omega > 70$
(ii) ${}_{40}p_0 = 0.6$
(iii) $\mathbf{E}[T] = 62$
(iv) $\mathbf{E}\big[\min(T,t)\big] = t - 0.005t^2$, $0 < t < 60$

Calculate the complete expectation of life at 40.

(A) 30 (B) 35 (C) 40 (D) 45 (E) 50

Exercises continue on the next page . . .

6.11. **[3-F00:4]** Mortality for Audra, age 25, follows $l_x = 50(100 - x)$, $0 \le x \le 100$.

If she takes up hot air ballooning for the coming year, her assumed mortality will be adjusted so that for the coming year only, she will have a constant force of mortality of 0.1.

Calculate the decrease in the 11-year temporary complete life expectancy for Audra if she takes up hot air ballooning.

(A) 0.10 (B) 0.35 (C) 0.60 (D) 0.80 (E) 1.00

6.12. **[3-F00:25]** Given:

(i) Superscripts M and N identify two forces of mortality and the curtate expectations of life calculated from them.

(ii) $\mu_{25+t}^N = \begin{cases} \mu_{25+t}^M + 0.1(1-t) & 0 \le t \le 1 \\ \mu_{25+t}^M & t > 1 \end{cases}$

(iii) $e_{25}^M = 10.0$

Calculate e_{25}^N.

(A) 9.2 (B) 9.3 (C) 9.4 (D) 9.5 (E) 9.6

6.13. You are given that $e_{35} = 49$ and $p_{35} = 0.995$.

If μ_x is doubled for $35 \le x \le 36$, what is the revised value of e_{35}?

6.14. **[CAS4-F82:33]** Which of the following statements is true concerning the inequality $e_{x+1} > e_x$?

(A) The inequality cannot be true.

(B) The inequality is true if and only if

$$e_{x+1} > \frac{p_x}{q_{x+1}}$$

(C) The inequality is true if and only if

$$e_{x+1} > \frac{p_x}{p_{x+1}q_{x+1}}$$

(D) The inequality is true if and only if

$$e_{x+1} > \frac{p_x + 1}{q_x}$$

(E) The inequality is true if and only if

$$e_{x+1} > \frac{p_x}{q_x}$$

6.15. **[CAS4A-F97:20]** (1 point) For a life age 50, the curtate expectation of life $e_{50} = 20$. For that same life, you are also given that $p_{50} = 0.97$.

Determine e_{51}.

(A) Less than 18.75
(B) At least 18.75, but less than 19.00
(C) At least 19.00, but less than 19.25
(D) At least 19.25, but less than 19.50
(E) At least 19.50

Exercises continue on the next page . . .

6.16. **[CAS4-S86:26]** (2 points) Consider a subgroup of lives that have been exposed to a certain disease. It is estimated that this subgroup will have a higher than normal rate of mortality for two years following exposure to this disease. The mortality rate is 10% higher than normal during the first year and 5% higher during the second year. After that the mortality rate returns to normal.

You are given:

(i) $q_x = 0.07$
(ii) $q_{x+1} = 0.10$
(iii) $q_{x+2} = 0.11$
(iv) $e_{x+3} = 5$

Calculate the reduction in curtate life expectancy, in years, for a person age (x) who has just been exposed to this disease.

(A) Less than 0.050
(B) At least 0.050, but less than 0.075
(C) At least 0.075, but less than 0.100
(D) At least 0.100, but less than 0.125
(E) At least 0.125

6.17. You are given

(i) $\mathring{e}_{40} = 35$
(ii) $\mathring{e}_{40:\overline{10}|} = 9$
(iii) ${}_{10}p_{40} = 0.85$
(iv) ${}_{t}p_{50} = 1 - 0.01t$ for $0 \le t \le 1$.

Improvements in mortality at age 50 cause ${}_{t}p_{50}$ to change to $1 - 0.009t$ for $0 \le t \le 1$.

Calculate the revised value of $\mathring{e}_{40}$.

6.18. You are given:

(i) $S_0(20) = 0.9$
(ii) $S_0(60) = y$
(iii) The survival distribution function is linear between ages 20 and 60.
(iv) $\mathring{e}_{20} = 60$
(v) $\mathring{e}_{60} = 25$

Determine y.

6.19. You are given:

(i) $\mathring{e}_{40} = 75$
(ii) $\mathring{e}_{60} = 70$
(iii) The force of mortality μ_x for $x \in [40, 60]$ is $1/(k - x)$ for some k.

Determine k.

6.20. Mortality follows Gompertz's law with $B = 0.001$ and $c = 1.05$.

You are given that $e_{40} = 34.97$.

Determine e_{41}.

Exercises continue on the next page . . .

6.21. Mortality follows Gompertz's law. You are given $e_{80} = 5.665$, $e_{81} = 5.362$ and $e_{82} = 5.071$. Determine e_{83}.

6.22. You are given:

(i) $e_{35.7} = 45$

(ii) ${}_{0.7}p_{35} = 0.99$

(iii) ${}_{0.3}p_{34.7} = 0.995$.

Calculate $e_{34.7}$.

Additional old CAS Exam 3/3L questions: F08:13, S10:1, S12:2

Additional old CAS Exam LC questions: S14:2

Solutions

6.1. Under Gompertz's law,

$$\begin{aligned} S_0(x) &= \exp\left(-\int_0^x Bc^t \, dt\right) \\ &= \exp\left(\frac{-B(c^x - 1)}{\ln c}\right) \end{aligned}$$

We set $S_0(40) = 0.9$ (the 90$^{\text{th}}$ percentile of the survival function is the 10$^{\text{th}}$ percentile of future lifetime at birth) and $S_0(80) = 0.3$.

$$\begin{aligned} \exp\left(\frac{-B(c^{40} - 1)}{\ln c}\right) &= 0.9 \\ \frac{B(c^{40} - 1)}{\ln c} &= -\ln 0.9 && (*) \\ \exp\left(\frac{-B(c^{80} - 1)}{\ln c}\right) &= 0.3 \\ \frac{B(c^{80} - 1)}{\ln c} &= -\ln 0.3 && (**) \end{aligned}$$

Dividing (*) into (**) and using $c^{80} - 1 = (c^{40} + 1)(c^{40} - 1)$,

$$\begin{aligned} \frac{c^{80} - 1}{c^{40} - 1} &= c^{40} + 1 = \frac{-\ln 0.3}{-\ln 0.9} = 11.427173 \\ c &= \sqrt[40]{11.427173 - 1} = \boxed{\mathbf{1.060362}} \end{aligned}$$

6.2. For this beta, ${}_tp_{30} = \sqrt{(90 - t)/90}$. Setting this equal to 0.5,

$$\begin{aligned} \left(\frac{90 - t}{90}\right)^{0.5} &= 0.5 \\ \frac{90 - t}{90} &= 0.5^2 = 0.25 \\ t &= 90 - 22.5 = \boxed{\mathbf{67.5}} \end{aligned}$$

6.3. The median future lifetime for (y) is the age z such that the probability of living to age z, given living to age y, is 0.5. Now, $S_0(z) = \frac{1}{1+z}$ and $S_0(y) = \frac{1}{1+y}$, so we want

$$\frac{1}{1+z} = 0.5\left(\frac{1}{1+y}\right)$$

Let's solve for z.

$$\begin{aligned}\frac{1}{1+x} &= \frac{1}{2(1+z)}\\ z &= 1+2y\end{aligned}$$

z is the median age at death for (y). Median future lifetime for (y) is $z - y = (1+2y) - y = \boxed{\mathbf{1+y}}$. **(A)**

This survival function is a Pareto. You can see why a Pareto distribution is not a plausible distribution for human life: the older (y) is, the longer (y)'s median future lifetime!

6.4.

I. We can compare the complements, ${}_{t+u}p_x$ and ${}_u p_{x+t}$. The former is the latter times ${}_t p_x$, and is therefore less than or equal to the latter. It follows the complements have the reverse relationship and I is true. ✓

II. ${}_{t|u}q_x = {}_t p_x \, {}_u q_{x+t}$, and ${}_t p_x \le 1$, so II is true. ✓

III. Both mean and median are equal to the midrange, so III is true. ✓

(D)

6.5. Survival is uniform with $\omega = 100$, so the median is the midrange, $\frac{100-10}{2} = \boxed{\mathbf{45}}$ **(C)**

6.6. The conditional survival function is

$$\begin{aligned}{}_t p_{20} &= \exp\left(-\int_0^t \sqrt{\frac{1}{80-20-u}}\,\mathrm{d}u\right)\\ &= \exp\left(2\sqrt{60-u}\right)\Big|_0^t\\ &= \exp\Big(2\big(\sqrt{60-t}-\sqrt{60}\big)\Big)\end{aligned}$$

We set this equal to 0.5 to get the median.

$$\begin{aligned}2\left(\sqrt{60-t}-\sqrt{60}\right) &= \ln 0.5\\ \sqrt{60-t} &= \sqrt{60}+\frac{\ln 0.5}{2} = 7.7460 - \frac{0.6931}{2} = 7.3994\\ 60-t &= 7.3994^2 = 54.751\\ t &= 60-54.751 = \boxed{\mathbf{5.249}} \qquad \textbf{(A)}\end{aligned}$$

6.7. The population has a mixture distribution; the probability of lifetime greater than x is $0.3e^{-0.2x} + 0.7e^{-0.1x}$. We want to set this equal to 0.25, so that there will be a 25% chance of living *longer* than x and therefore a 75% chance of living less. Let $y = e^{-0.1x}$. Then we have

$$0.3y^2 + 0.7y - 0.25 = 0$$

$$\begin{aligned}
y &= \frac{-0.7+\sqrt{0.49+0.3}}{0.6} = 0.31470\\
e^{-0.1x} &= 0.31470\\
x &= -10\ln 0.31470 = \boxed{11.5614} \qquad \textbf{(D)}
\end{aligned}$$

6.8. Since the 60th percentile is the t for which $F(t) = 0.6$, we set $F_{45}(t) = 0.6$, or in other words ${}_tp_{45} = 0.4$.

$$\begin{aligned}
\exp\left(-0.0003(1.065^{45})\left(\frac{1.065^t-1}{\ln 1.065}\right)\right) &= 0.4\\
0.0003(1.065^{45})\left(\frac{1.065^t-1}{\ln 1.065}\right) &= -\ln 0.4\\
1.065^t - 1 &= \frac{(-\ln 0.4)(\ln 1.065)}{0.0003(1.065^{45})} = 11.30698\\
t &= \frac{\ln 12.30698}{\ln 1.065} = \boxed{39.86} \qquad \textbf{(E)}
\end{aligned}$$

6.9. We will do three recursions, one age at a time. Since $e_x = p_x(1+e_{x+1})$, it follows that $e_{x+1} = (e_x/p_x) - 1$.

$$\begin{aligned}
e_{66} &= 24/0.99 - 1 = 23.24242\\
e_{67} &= 23.24242/0.985 - 1 = 22.59637\\
e_{68} &= 22.59637/0.98 - 1 = \boxed{22.0575}
\end{aligned}$$

6.10. The complete expectation of life is the complete expectation bounded by t plus the probability of survival to t times the complete expectation of life after t years (formula (6.1)).

$$\begin{aligned}
\mathbf{E}[T] &= \mathbf{E}\big[\min(T,40)\big] + \mathring{e}_{40}\,({}_{40}p_0)\\
62 &= \big(40 - 0.005(40^2)\big) + \mathring{e}_{40}(0.6)\\
&= 32 + \mathring{e}_{40}(0.6)\\
\mathring{e}_{40} &= \frac{62-32}{0.6} = \boxed{50} \qquad \textbf{(E)}
\end{aligned}$$

6.11. The original 11-year temporary complete life expectancy is computed using equation (5.12):

$$\begin{aligned}
\mathring{e}_{25:\overline{11}|} &= {}_{11}p_{25}(11) + {}_{11}q_{25}(5.5)\\
&= \frac{64(11)}{75} + \frac{11(5.5)}{75} = 10.193333
\end{aligned}$$

The 10-year temporary complete life expectancy for (26) is

$$\begin{aligned}
\mathring{e}_{26:\overline{10}|} &= {}_{10}p_{26}(10) + {}_{10}q_{26}(5)\\
&= \frac{64(10)}{74} + \frac{10(5)}{74} = 9.324324
\end{aligned}$$

We recursively develop the modified $\mathring{e}_{25:\overline{11}|}$ using equation (6.1). We'll use a prime to denote the modified functions.

$$\mathring{e}'_{25:\overline{11}|} = \mathring{e}'_{25:\overline{1}|} + p'_{25}\,\mathring{e}_{26:\overline{10}|}$$

$$\begin{aligned} p'_{25} &= e^{-0.1} \\ \mathring{e}'_{25:\overline{1}|} &= \int_0^1 e^{-0.1t}\,\mathrm{d}t = 10\left(1 - e^{-0.1}\right) \\ \mathring{e}'_{25:\overline{11}|} &= 10\left(1 - e^{-0.1}\right) + e^{-0.1}(9.324324) \\ &= 0.951626 + 8.436997 = 9.388623 \end{aligned}$$

The difference in mortality is $10.193333 - 9.388623 = \boxed{\mathbf{0.8047}}$ **(D)**

6.12. As discussed in Table 3.1 at the bottom, adding an expression to μ_x multiplies survival probabilities by the exponential of negative its integral, so

$$\begin{aligned} p_{25}^N &= p_{25}^M \exp\left(-\int_0^1 0.1(1-t)\mathrm{d}t\right) \\ &= p_{25}^M \exp\left(-0.05(1-t)^2\Big|_0^1\right) \\ &= e^{-0.05} p_{25}^M \end{aligned}$$

Since ${}_tp_{26}^N = {}_tp_{26}^M$ for all t, it follows that $e_{26}^N = e_{26}^M$. We use recursive formula (6.4).

$$\begin{aligned} e_{25}^N &= p_{25}^N(1 + e_{26}^N) \\ &= e^{-0.05} p_{25}^M (1 + e_{26}^M) \\ &= e^{-0.05} e_{25}^M \\ &= 10e^{-0.05} = \boxed{\mathbf{9.5123}} \qquad \textbf{(D)} \end{aligned}$$

6.13. As discussed at the bottom of Table 3.1, doubling μ squares p_{35}. Using primes for revised values, $p'_{35} = 0.995^2$. Then using the recursion of formula (6.4)

$$1 + e_{36} = e_{35}/p_{35} = e_{35}/0.995$$

and since $e_{36} = e'_{36}$,

$$e'_{35} = p'_{35}(1 + e_{36}) = 0.995^2(1 + e_{36}) = \frac{0.995^2}{0.995} e_{35}$$

so we conclude

$$e'_{35} = 49(0.995) = \boxed{\mathbf{48.755}}$$

6.14. By equation (6.4)

$$e_x = p_x + p_x e_{x+1}$$

So we want

$$\begin{aligned} e_{x+1} &> p_x + p_x\, e_{x+1} = p_x + e_{x+1} - q_x\, e_{x+1} \\ 0 &> p_x - q_x\, e_{x+1} \\ e_{x+1} &> \frac{p_x}{q_x} \qquad \textbf{(E)} \end{aligned}$$

6.15. Using formula (6.4),

$$\begin{aligned} e_{50} &= p_{50} + p_{50}\, e_{51} \\ 20 &= 0.97 + 0.97 e_{51} \\ e_{51} &= \frac{20}{0.97} - 1 = \boxed{\mathbf{19.6186}} \qquad \textbf{(E)} \end{aligned}$$

6.16. *Note that mortality **rate** refers to q_x, not μ_x.* The latter, μ_x, would be called *force* of mortality.

Using primes to indicate modified functions for the subgroup of lives exposed to the disease,

$$e_x = \sum_{k=1}^{\infty} {}_kp_x = p_x + {}_2p_x + {}_3p_x(1 + e_{x+3}) = p_x + {}_2p_x + 6\,{}_3p_x$$

so the difference between the two life expectancies is

$$p_x - p'_x + {}_2p_x - {}_2p'_x + 6({}_3p_x - {}_3p'_x)$$

Let's calculate the needed survival probabilities.

$$q'_x = 0.07(1.1) = 0.077 \qquad p'_x = 0.923$$
$$q'_{x+1} = 0.10(1.05) = 0.105 \qquad p'_{x+1} = 0.895$$

$$p_x = 0.93 \qquad p'_x = 0.923 \qquad p_x - p'_x = 0.007$$
$${}_2p_x = (0.93)(0.9) = 0.837 \qquad {}_2p'_x = (0.923)(0.895) = 0.826085 \qquad {}_2p_x - {}_2p'_x = 0.010915$$
$${}_3p_x = (0.837)(0.89) = 0.74493 \qquad {}_3p'_x = (0.826085)(0.89) = 0.735216 \qquad {}_3p_x - {}_3p'_x = 0.009714$$

The answer is $0.007 + 0.010915 + 6(0.009714) = \boxed{0.07620}$. **(C)**

6.17. First we use the recursive formula to calculate $\mathring{e}_{50}$.

$$\mathring{e}_{40} = \mathring{e}_{40:\overline{10}|} + {}_{10}p_{40}\,\mathring{e}_{50}$$
$$35 = 9 + 0.85\mathring{e}_{50}$$
$$\mathring{e}_{50} = \frac{26}{0.85} = 30.588235$$

Then we use the recursive formula to calculate $\mathring{e}_{51}$. By substituting $t = 1$ into condition (iv), w see that $p_{50} = 0.99$. Since deaths are uniformly distributed over age 50, by either using the trapezoidal rule or the formula $\mathring{e}_{50:\overline{1}|} = 0.5q_{50} + p_{50}$ (equation (5.12)), we get $\mathring{e}_{50:\overline{1}|} = 0.5(1 + 0.99) = 0.995$. Then

$$\mathring{e}_{50} = \mathring{e}_{50:\overline{1}|} + p_{50}\mathring{e}_{51}$$
$$30.588235 = 0.995 + 0.99\mathring{e}_{51}$$
$$\mathring{e}_{51} = 29.892157$$

We use primes for the revised functions. Substituting $t = 1$ into ${}_tp'_{50} = 1 - 0.009t$, we get $p'_{50} = 0.991$. Using the trapezoidal rule or equation (5.12), $\mathring{e}'_{50:\overline{1}|} = 0.5(1 + 0.991) = 0.9955$.

$$\mathring{e}'_{50} = 0.9955 + 0.991(29.892157) = 30.618627$$
$$\mathring{e}'_{40} = 9 + 0.85(30.618627) = \boxed{35.025833}$$

6.18. Split the universe into two groups, the ones that survive to age 60 and the ones who don't. The ones who survive to age 60 have an expected lifetime of 40 plus $\mathring{e}_{60}$, or 65. The ones who don't survive to age 60 have an expected lifetime of 20, since survival is uniform between ages 20 and 60. Expected lifetime at 20 is the weighted average of the expected lifetime of these two groups:

$$\mathring{e}_{20} = {}_{40}p_{20}(65) + (1 - {}_{40}p_{20})(20)$$
$$60 = 20 + 45\,{}_{40}p_{20}$$
$${}_{40}p_{20} = \frac{40}{45} = \frac{8}{9}$$

But ${}_{40}p_{20} = S_0(60)/S_0(20)$, so it follows that $S_0(60) = (8/9)S_0(20) = \boxed{0.8}$.

6.19. Note that mortality has a uniform distribution between 40 and 60, since μ_x has the form $1/(\omega - x)$ in that interval. (This is *not* saying that mortality is uniform globally; it may be non-uniform above 60. All we're saying is that ${}_tq_{40} = tq_{40}$ for $t \le 20$.) To show this rigorously, for $0 \le t \le 20$:

$$\begin{aligned}
{}_tp_{40} &= \exp\left(-\int_0^t \mu_{40+s}\,\mathrm{d}s\right)\\
&= \exp\left(-\int_0^t \frac{\mathrm{d}s}{k-40-s}\right)\\
&= \exp\big(\ln(k-40-t) - \ln(k-40)\big)\\
&= \frac{k-40-t}{k-40}\\
{}_tq_{40} &= \frac{t}{k-40}
\end{aligned}$$

This is a linear function of t. Whenever we have uniform mortality, the double expectation theorem is very helpful for calculating expectation:

$$\begin{aligned}
\mathring{e}_{40} &= \Pr(T_{40} > 60)\,\mathbf{E}[T_{40} \mid T_{40} > 60] + \Pr(T_{40} \le 60)\,\mathbf{E}[T_{40} \mid T_{40} \le 60]\\
75 &= {}_{20}p_{40}(20+70) + {}_{20}q_{40}\,\mathbf{E}[T_{40} \mid T_{40} \le 60]
\end{aligned}$$

Notice that $\mathbf{E}[T_{40} \mid T_{40} > 60]$ is 20 years (from age 40 to age 60) plus the expected future lifetime of a 60-year old, or $20 + \mathring{e}_{60} = 20 + 70 = 90$. Since T_{40} is uniform between 0 and 20, its average, given that it is in that range, is the midpoint, or 10. So the previous equation becomes

$$\begin{aligned}
75 &= (1 - {}_{20}q_{40})(90) + {}_{20}q_{40}(10) = 90 - 80\,{}_{20}q_{40}\\
{}_{20}q_{40} &= \frac{15}{80} = \frac{3}{16}\\
\frac{20}{k-40} &= \frac{3}{16}\\
3k - 120 &= 320\\
k &= \frac{440}{3} = \boxed{\mathbf{146\tfrac{2}{3}}}
\end{aligned}$$

6.20. Let's calculate p_{40}.

$$p_{40} = \exp\left(-\frac{0.001(1.05^{40})(0.05)}{\ln 1.05}\right) = 0.992811$$

Then from the recursive formula,

$$\begin{aligned}
e_{40} &= p_{40}(1 + e_{41})\\
34.97 &= 0.992811(1 + e_{41})\\
e_{41} &= \frac{34.97}{0.992811} - 1 = \boxed{\mathbf{34.22}}
\end{aligned}$$

6.21. Back out p_{80} and p_{81} from the recursive formula.

$$\begin{aligned}
e_{80} &= p_{80}(1 + e_{81})\\
p_{80} &= \frac{5.665}{6.362} = 0.890443
\end{aligned}$$

$$p_{81} = \frac{5.362}{6.071} = 0.883215$$

Now back out the c parameter of Gompertz's law.

$$\begin{aligned}\frac{Bc^{80}(c-1)}{\ln c} &= -\ln 0.890443\\ \frac{Bc^{81}(c-1)}{\ln c} &= -\ln 0.883215\\ c &= \frac{\ln 0.883215}{\ln 0.890443} = 1.070240\end{aligned}$$

We need $Bc^{82}/(c-1)$, but that is c times $Bc^{81}/(c-1)$

$$Bc^{82} = (-\ln 0.883215)(1.070240) = 0.132909$$

Now we can calculate p_{82}.

$$\begin{aligned}p_{82} &= \exp(-0.132909) = 0.87554\\ e_{83} &= \frac{e_{82}}{p_{82}} - 1 = \frac{5.071}{0.87554} - 1 = \boxed{\mathbf{4.7918}}\end{aligned}$$

6.22. By the recursive formula,

$$e_{34.7} = p_{34.7}(1 + e_{35.7}) = (0.99)(0.995)(1 + 45) = \boxed{\mathbf{45.3123}}$$

Quiz Solutions

6-1. The third quartile is the 75th percentile. The survival probability is

$$\begin{aligned}{}_tp_x &= \exp\left(-\int_0^t \frac{\mathrm{d}u}{u+50}\right)\\ &= \exp\bigl(-\ln(t+50) + \ln(50)\bigr) = \frac{50}{50+t}\end{aligned}$$

This equals 0.25 at:

$$\begin{aligned}\frac{50}{50+t} &= 0.25\\ t &= \boxed{\mathbf{150}}\end{aligned}$$

6-2. Since $e_0 = p_0(1 + e_1)$, we have $e_1 = 70/p_0 - 1 > 70$, so

$$\begin{aligned}\frac{70}{p_0} &> 71\\ p_0 &< \boxed{\mathbf{\tfrac{70}{71}}}\end{aligned}$$

Lesson 7

Survival Distributions: Fractional Ages

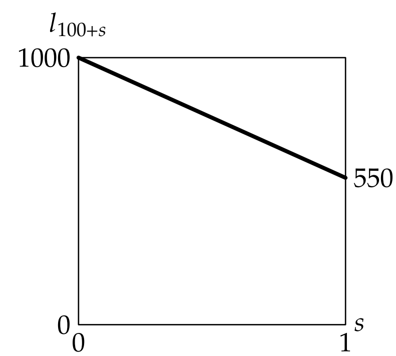
Reading: *Actuarial Mathematics for Life Contingent Risks* 2nd edition 3.2

Life tables list mortality rates (q_x) or lives (l_x) for integral ages only. Often, it is necessary to determine lives at fractional ages (like $l_{x+0.5}$ for x an integer) or mortality rates for fractions of a year. We need some way to interpolate between ages.

7.1 Uniform distribution of deaths

The easiest interpolation method is linear interpolation, or uniform distribution of deaths between integral ages (UDD). This means that the number of lives at age $x+s$, $0 \le s \le 1$, is a weighted average of the number of lives at age x and the number of lives at age $x+1$:

$$l_{x+s} = (1-s)l_x + sl_{x+1} = l_x - sd_x \tag{7.1}$$



The graph of l_{x+s} is a straight line between $s=0$ and $s=1$ with slope $-d_x$. The graph at the right portrays this for a mortality rate $q_{100} = 0.45$ and $l_{100} = 1000$.

Contrast UDD with an assumption of a uniform survival function. If age at death is uniformly distributed, then l_x as a function of x is a straight line. If UDD is assumed, l_x is a straight line between integral ages, but the slope may vary for different ages. Thus if age at death is uniformly distributed, UDD holds at all ages, but not conversely.

Using l_{x+s}, we can compute ${}_sq_x$:

$$\begin{aligned} {}_sq_x &= 1 - {}_sp_x \\ &= 1 - \frac{l_{x+s}}{l_x} = 1 - (1 - sq_x) = sq_x \end{aligned} \tag{7.2}$$

That is one of the most important formulas, so let's state it again:

$$\boxed{{}_sq_x = sq_x} \tag{7.2}$$

More generally, for $0 \le s+t \le 1$,

$$\begin{aligned} {}_sq_{x+t} &= 1 - {}_sp_{x+t} = 1 - \frac{l_{x+s+t}}{l_{x+t}} \\ &= 1 - \frac{l_x - (s+t)d_x}{l_x - td_x} = \frac{sd_x}{l_x - td_x} = \frac{sq_x}{1 - tq_x} \end{aligned} \tag{7.3}$$

where the last equation was obtained by dividing numerator and denominator by l_x. The important point to pick up is that while ${}_sq_x$ is the proportion of the year s times q_x, the corresponding concept at age $x+t$, ${}_sq_{x+t}$, is *not* sq_x, but is in fact higher than sq_x. The *number* of lives dying in any amount of time is constant, and since there are fewer and fewer lives as the year progresses, the *rate* of death is in fact increasing over the year. The numerator of ${}_sq_{x+t}$ is the proportion of the year being measured s times the death rate, but then this must be divided by 1 minus the proportion of the year that elapsed before the start of measurement.

For most problems involving death probabilities, it will suffice if you remember that l_{x+s} is linearly interpolated. It often helps to create a life table with an arbitrary radix. Try working out the following example before looking at the answer.

Example 7A You are given:

(i) $q_x = 0.1$

(ii) Uniform distribution of deaths between integral ages is assumed.

Calculate ${}_{1/2}q_{x+1/4}$.

Answer: Let $l_x = 1$. Then $l_{x+1} = l_x(1 - q_x) = 0.9$ and $d_x = 0.1$. Linearly interpolating,

$$l_{x+1/4} = l_x - \tfrac{1}{4}d_x = 1 - \tfrac{1}{4}(0.1) = 0.975$$
$$l_{x+3/4} = l_x - \tfrac{3}{4}d_x = 1 - \tfrac{3}{4}(0.1) = 0.925$$
$${}_{1/2}q_{x+1/4} = \frac{l_{x+1/4} - l_{x+3/4}}{l_{x+1/4}} = \frac{0.975 - 0.925}{0.975} = \boxed{0.051282}$$

You could also use equation (7.3) to work this example. □

Example 7B For two lives age (x) with independent future lifetimes, ${}_{k|}q_x = 0.1(k+1)$ for $k = 0, 1, 2$. Deaths are uniformly distributed between integral ages.

Calculate the probability that both lives will survive 2.25 years.

Answer: Since the two lives are independent, the probability of both surviving 2.25 years is the square of ${}_{2.25}p_x$, the probability of one surviving 2.25 years. If we let $l_x = 1$ and use $d_{x+k} = l_x\,{}_{k|}q_x$, we get

$$q_x = 0.1(1) = 0.1 \qquad l_{x+1} = 1 - d_x = 1 - 0.1 = 0.9$$
$${}_{1|}q_x = 0.1(2) = 0.2 \qquad l_{x+2} = 0.9 - d_{x+1} = 0.9 - 0.2 = 0.7$$
$${}_{2|}q_x = 0.1(3) = 0.3 \qquad l_{x+3} = 0.7 - d_{x+2} = 0.7 - 0.3 = 0.4$$

Then linearly interpolating between l_{x+2} and l_{x+3}, we get

$$l_{x+2.25} = 0.7 - 0.25(0.3) = 0.625$$
$${}_{2.25}p_x = \frac{l_{x+2.25}}{l_x} = 0.625$$

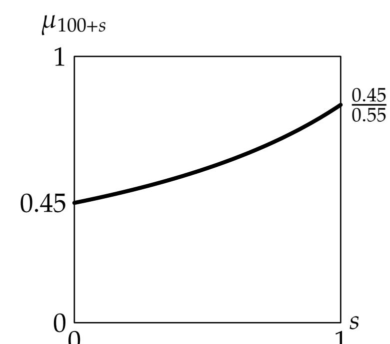
Squaring, the answer is $0.625^2 = \boxed{0.390625}$. □

The probability density function of T_x, ${}_sp_x\,\mu_{x+s}$, is the constant q_x, the derivative of the conditional cumulative distribution function ${}_sq_x = sq_x$ with respect to s. That is another important formula, since the density is needed to compute expected values, so let's repeat it:

$$\boxed{{}_sp_x\,\mu_{x+s} = q_x} \tag{7.4}$$

It follows that the force of mortality is q_x divided by $1 - sq_x$:

$$\mu_{x+s} = \frac{q_x}{{}_sp_x} = \frac{q_x}{1 - sq_x} \tag{7.5}$$



The force of mortality increases over the year, as illustrated in the graph for $q_{100} = 0.45$ to the right.

Quiz 7-1 You are given:

(i) $\mu_{50.4} = 0.01$

(ii) Deaths are uniformly distributed between integral ages.

Calculate ${}_{0.6}q_{50.4}$.

Complete Expectation of Life Under UDD

Under uniform distribution of deaths between integral ages, if the complete future lifetime random variable T_x is written as $T_x = K_x + R_x$, where K_x is the curtate future lifetime and R_x is the fraction of the last year lived, then K_x and R_x are independent, and R_x is uniform on $[0,1)$. If uniform distribution of deaths is not assumed, K_x and R_x are usually not independent. Since R_x is uniform on $[0,1)$, $\mathbf{E}[R_x] = \frac{1}{2}$ and $\mathrm{Var}(R_x) = \frac{1}{12}$. It follows from $\mathbf{E}[R_x] = \frac{1}{2}$ that

$$\mathring{e}_x = e_x + \tfrac{1}{2} \tag{7.6}$$

Let's discuss temporary complete life expectancy. You can always evaluate the temporary complete expectancy, whether or not UDD is assumed, by integrating ${}_tp_x$, as indicated by formula (5.6) on page 80. For UDD, ${}_tp_x$ is linear between integral ages. Therefore, a rule we learned in Lesson 5 applies for all integral x:

$$\mathring{e}_{x:\overline{1}|} = p_x + 0.5q_x \tag{5.13}$$

This equation will be useful. In addition, the method for generating this equation can be used to work out questions involving temporary complete life expectancies for short periods. The following example illustrates this. This example will be reminiscent of calculating temporary complete life expectancy for uniform mortality.

Example 7C You are given

(i) $q_x = 0.1$.

(ii) Deaths are uniformly distributed between integral ages.

Calculate $\mathring{e}_{x:\overline{0.4}|}$.

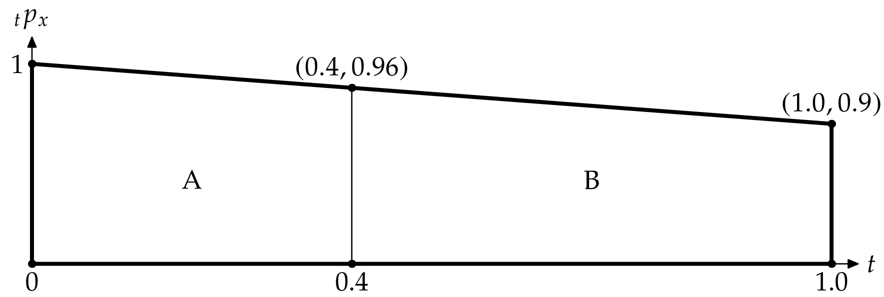
Answer: We will discuss two ways to solve this: an algebraic method and a geometric method.

The algebraic method is based on the double expectation theorem, equation (1.11). It uses the fact that *for a uniform distribution, the mean is the midpoint*. If deaths occur uniformly between integral ages, then those who die within a period contained within a year survive half the period on the average.

In this example, those who die within 0.4 survive an average of 0.2. Those who survive 0.4 survive an average of 0.4 of course. The temporary life expectancy is the weighted average of these two groups, or ${}_{0.4}q_x(0.2) + {}_{0.4}p_x(0.4)$. This is:

$$\begin{aligned}
{}_{0.4}q_x &= (0.4)(0.1) = 0.04\\
{}_{0.4}p_x &= 1 - 0.04 = 0.96\\
\mathring{e}_{x:\overline{0.4}|} &= 0.04(0.2) + 0.96(0.4) = \boxed{0.392}
\end{aligned}$$

An equivalent geometric method, the trapezoidal rule, is to draw the ${}_tp_x$ function from 0 to 0.4. The integral of ${}_tp_x$ is the area under the line, which is the area of a trapezoid: the average of the heights times the width. The following is the graph (not drawn to scale):



Trapezoid A is the area we are interested in. Its area is $\frac{1}{2}(1+0.96)(0.4) = \boxed{\mathbf{0.392}}$. □

Quiz 7-2 As in Example 7C, you are given

(i) $q_x = 0.1$.

(ii) Deaths are uniformly distributed between integral ages.

Calculate $\mathring{e}_{x+0.4:\overline{0.6}|}$.

Let's now work out an example in which the duration crosses an integral boundary.

Example 7D You are given:

(i) $q_x = 0.1$

(ii) $q_{x+1} = 0.2$

(iii) Deaths are uniformly distributed between integral ages.

Calculate $\mathring{e}_{x+0.5:\overline{1}|}$.

Answer: Let's start with the algebraic method. Since the mortality rate changes at $x+1$, we must split the group into those who die before $x+1$, those who die afterwards, and those who survive. Those who die before $x+1$ live 0.25 on the average since the period to $x+1$ is length 0.5. Those who die after $x+1$ live between 0.5 and 1 years; the midpoint of 0.5 and 1 is 0.75, so they live 0.75 years on the average. Those who survive live 1 year.

Now let's calculate the probabilities.

$$
\begin{aligned}
{}_{0.5}q_{x+0.5} &= \frac{0.5(0.1)}{1-0.5(0.1)} = \frac{5}{95} \\
{}_{0.5}p_{x+0.5} &= 1-\frac{5}{95} = \frac{90}{95} \\
{}_{0.5|0.5}q_{x+0.5} &= \left(\frac{90}{95}\right)(0.5(0.2)) = \frac{9}{95} \\
{}_{1}p_{x+0.5} &= 1-\frac{5}{95}-\frac{9}{95} = \frac{81}{95}
\end{aligned}
$$

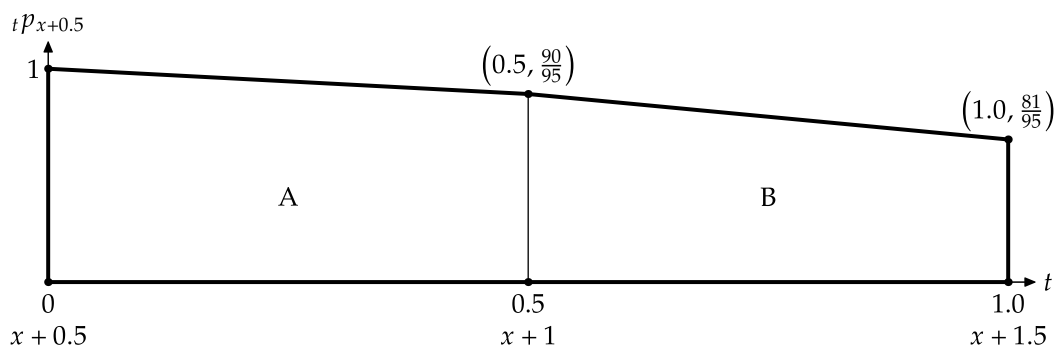
These probabilities could also be calculated by setting up an l_x table with radix 100 at age x and interpolating within it to get $l_{x+0.5}$ and $l_{x+1.5}$. Then

$$\begin{aligned}
l_{x+1} &= 0.9l_x = 90\\
l_{x+2} &= 0.8l_{x+1} = 72\\
l_{x+0.5} &= 0.5(90+100) = 95\\
l_{x+1.5} &= 0.5(72+90) = 81\\
{}_{0.5}q_{x+0.5} &= 1-\frac{90}{95} = \frac{5}{95}\\
{}_{0.5|0.5}q_{x+0.5} &= \frac{90-81}{95} = \frac{9}{95}\\
{}_1p_{x+0.5} &= \frac{l_{x+1.5}}{l_{x+0.5}} = \frac{81}{95}
\end{aligned}$$

Either way, we're now ready to calculate $\mathring{e}_{x+0.5:\overline{1}|}$.

$$\mathring{e}_{x+0.5:\overline{1}|} = \frac{5(0.25)+9(0.75)+81(1)}{95} = \boxed{\frac{89}{95}}$$

For the geometric method we draw the following graph:



The heights at $x+1$ and $x+1.5$ are as we computed above. Then we compute each area separately. The area of A is $\frac{1}{2}\left(1+\frac{90}{95}\right)(0.5) = \frac{185}{95(4)}$. The area of B is $\frac{1}{2}\left(\frac{90}{95}+\frac{81}{95}\right)(0.5) = \frac{171}{95(4)}$. Adding them up, we get $\frac{185+171}{95(4)} = \boxed{\frac{89}{95}}$. □

Quiz 7-3 The probability that a battery fails by the end of the k^{th} month is given in the following table:

k	Probability of battery failure by the end of month k
1	0.05
2	0.20
3	0.60

Between integral months, time of failure for the battery is uniformly distributed.
Calculate the expected amount of time the battery survives within 2.25 months.

To calculate $\mathring{e}_{x:\overline{n}|}$ in terms of $e_{x:\overline{n}|}$ when x and n are both integers, note that those who survive n years contribute the same to both. Those who die contribute an average of $\frac{1}{2}$ more to $\mathring{e}_{x:\overline{n}|}$ since they die on the average in the middle of the year. Thus the difference is $\frac{1}{2}{}_nq_x$:

$$\mathring{e}_{x:\overline{n}|} = e_{x:\overline{n}|} + 0.5{}_nq_x \tag{7.7}$$

Example 7E You are given:

(i) $q_x = 0.01$ for $x = 50, 51, \ldots, 59$.

(ii) Deaths are uniformly distributed between integral ages.

Calculate $\mathring{e}_{50:\overline{10}|}$.

Answer: As we just said, $\mathring{e}_{50:\overline{10}|} = e_{50:\overline{10}|} + 0.5{}_{10}q_{50}$. The first summand, $e_{50:\overline{10}|}$, is the sum of ${}_kp_{50} = 0.99^k$ for $k = 1, \ldots, 10$. This sum is a geometric series:

$$e_{50:\overline{10}|} = \sum_{k=1}^{10} 0.99^k = \frac{0.99 - 0.99^{11}}{1 - 0.99} = 9.46617$$

The second summand, the probability of dying within 10 years is ${}_{10}q_{50} = 1 - 0.99^{10} = 0.095618$. Therefore

$$\mathring{e}_{50:\overline{10}|} = 9.46617 + 0.5(0.095618) = \boxed{9.51398}$$

□

7.2 Constant force of mortality

The constant force of mortality interpolation method sets μ_{x+s} equal to a constant for x an integral age and $0 < s \leq 1$. Since $p_x = \exp\left(-\int_0^1 \mu_{x+s}\,ds\right)$ and $\mu_{x+s} = \mu$ is constant,

$$p_x = e^{-\mu} \tag{7.8}$$

$$\mu = -\ln p_x \tag{7.9}$$

Therefore

$${}_sp_x = e^{-\mu s} = (p_x)^s \tag{7.10}$$

In fact, ${}_sp_{x+t}$ is independent of t for $0 \leq t \leq 1 - s$.

$${}_sp_{x+t} = (p_x)^s \tag{7.11}$$

for any $0 \leq t \leq 1 - s$. Figure 7.1 shows l_{100+s} and μ_{100+s} for $l_{100} = 1000$ and $q_{100} = 0.45$ if constant force of mortality is assumed.

Contrast constant force of mortality between integral ages to global constant force of mortality, which was introduced in Subsection 4.2.1. The method discussed here allows μ_x to vary for different integers x.

We will now repeat some of the earlier examples but using constant force of mortality.

Example 7F You are given:

(i) $q_x = 0.1$

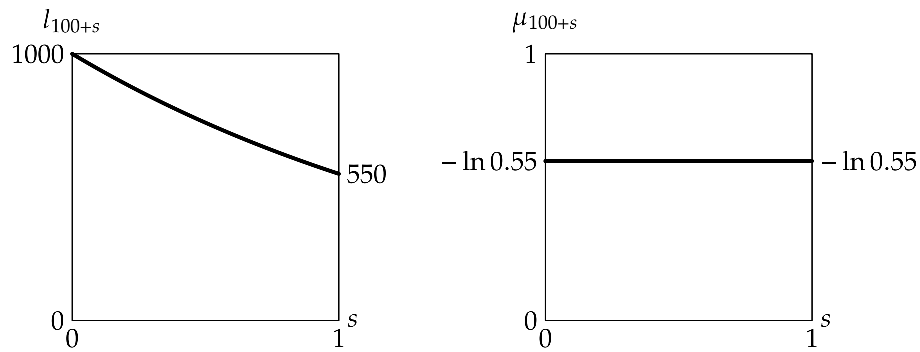
(ii) The force of mortality is constant between integral ages.

Calculate ${}_{1/2}q_{x+1/4}$.

Answer:

$${}_{1/2}q_{x+1/4} = 1 - {}_{1/2}p_{x+1/4} = 1 - p_x^{1/2} = 1 - 0.9^{1/2} = 1 - 0.948683 = \boxed{0.051317}$$

□



(a) l_{100+s} (b) μ_{100+s}

Figure 7.1: Example of constant force of mortality

Example 7G You are given:

(i) $q_x = 0.1$

(ii) $q_{x+1} = 0.2$

(iii) The force of mortality is constant between integral ages.

Calculate $\mathring{e}_{x+0.5:\overline{1}|}$.

Answer: We calculate $\int_0^1 {}_tp_{x+0.5}\,dt$. We split this up into two integrals, one from 0 to 0.5 for age x and one from 0.5 to 1 for age $x+1$. The first integral is

$$\int_0^{0.5} {}_tp_{x+0.5}\,dt = \int_0^{0.5} p_x^t\,dt = \int_0^{0.5} 0.9^t\,dt = -\frac{1-0.9^{0.5}}{\ln 0.9} = 0.487058$$

For $t > 0.5$,

$${}_tp_{x+0.5} = {}_{0.5}p_{x+0.5}\;{}_{t-0.5}p_{x+1} = 0.9^{0.5}\;{}_{t-0.5}p_{x+1}$$

so the second integral is

$$0.9^{0.5}\int_{0.5}^{1} {}_{t-0.5}p_{x+1}\,dt = 0.9^{0.5}\int_0^{0.5} 0.8^t\,dt = -\left(0.9^{0.5}\right)\left(\frac{1-0.8^{0.5}}{\ln 0.8}\right) = (0.948683)(0.473116) = 0.448837$$

The answer is $\mathring{e}_{x+0.5:\overline{1}|} = 0.487058 + 0.448837 = \boxed{0.935895}$. □

Although constant force of mortality is not used as often as UDD, it can be useful for simplifying formulas under certain circumstances. Calculating the expected present value of an insurance where the death benefit within a year follows an exponential pattern (this can happen when the death benefit is the discounted present value of something) may be easier with constant force of mortality than with UDD.

The formulas for this lesson are summarized in Table 7.1.

Table 7.1: Summary of formulas for fractional ages

Function	Uniform distribution of deaths	Constant force of mortality
l_{x+s}	$l_x - sd_x$	$l_x\, p_x^s$
${}_sq_x$	sq_x	$1 - p_x^s$
${}_sp_x$	$1 - sq_x$	p_x^s
${}_sq_{x+t}$	$sq_x/(1 - tq_x)$	$1 - p_x^s$
μ_{x+s}	$q_x/(1 - sq_x)$	$-\ln p_x$
${}_sp_x\, \mu_{x+s}$	q_x	$-p_x^s \ln p_x$
$\mathring{e}_x$	$e_x + 0.5$	
$\mathring{e}_{x:\overline{n}\rceil}$	$e_{x:\overline{n}\rceil} + 0.5\, {}_nq_x$	
$\mathring{e}_{x:\overline{1}\rceil}$	$p_x + 0.5q_x$	

Exercises

Uniform distribution of death

7.1. [**CAS4-S85:16**] (1 point) Deaths are uniformly distributed between integral ages.

Which of the following represents ${}_{3/4}p_x + \frac{1}{2}\,{}_{1/2}p_x\, \mu_{x+1/2}$?

(A) ${}_{3/4}p_x$ (B) ${}_{3/4}q_x$ (C) ${}_{1/2}p_x$ (D) ${}_{1/2}q_x$ (E) ${}_{1/4}p_x$

7.2. [Based on **150-S88:25**] You are given:

(i) ${}_{0.25}q_{x+0.75} = 3/31$.
(ii) Mortality is uniformly distributed within age x.

Calculate q_x.

Use the following information for questions 7.3 and 7.4:

You are given:
(i) Deaths are uniformly distributed between integral ages.
(ii) $q_x = 0.10$.
(iii) $q_{x+1} = 0.15$.

7.3. Calculate ${}_{1/2}q_{x+3/4}$.

7.4. Calculate ${}_{0.3|0.5}q_{x+0.4}$.

Exercises continue on the next page . . .

7.5. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) Mortality follows the Illustrative Life Table.

Calculate the median future lifetime for (45.5).

7.6. **[160-F90:5]** You are given:

(i) A survival distribution is defined by

$$l_x = 1000\left(1 - \left(\frac{x}{100}\right)^2\right),\ 0 \le x \le 100.$$

(ii) μ_x denotes the actual force of mortality for the survival distribution.
(iii) μ_x^L denotes the approximation of the force of mortality based on the uniform distribution of deaths assumption for l_x, $50 \le x < 51$.

Calculate $\mu_{50.25} - \mu_{50.25}^L$.

(A) −0.00016 (B) −0.00007 (C) 0 (D) 0.00007 (E) 0.00016

7.7. A survival distribution is defined by

(i) $S_0(k) = 1/(1 + 0.01k)^4$ for k a non-negative integer.
(ii) Deaths are uniformly distributed between integral ages.

Calculate ${}_{0.4}q_{20.2}$.

7.8. [Based on **150-S89:15**] You are given:

(i) Deaths are uniformly distributed over each year of age.
(ii)

x	l_x
35	100
36	99
37	96
38	92
39	87

Which of the following are true?

I. ${}_{1|2}q_{36} = 0.091$

II. $\mu_{37.5} = 0.043$

III. ${}_{0.33}q_{38.5} = 0.021$

(A) I and II only (B) I and III only (C) II and III only (D) I, II and III
(E) The correct answer is not given by (A) , (B) , (C) , or (D) .

Exercises continue on the next page . . .

7.9. **[150-82-94:5]** You are given:

(i) Deaths are uniformly distributed over each year of age.
(ii) ${}_{0.75}p_x = 0.25$.

Which of the following are true?

I. ${}_{0.25}q_{x+0.5} = 0.5$
II. ${}_{0.5}q_x = 0.5$
III. $\mu_{x+0.5} = 0.5$

(A) I and II only (B) I and III only (C) II and III only (D) I, II and III
(E) The correct answer is not given by (A) , (B) , (C) , or (D) .

7.10. **[3-S00:12]** For a certain mortality table, you are given:

(i) $\mu_{80.5} = 0.0202$
(ii) $\mu_{81.5} = 0.0408$
(iii) $\mu_{82.5} = 0.0619$
(iv) Deaths are uniformly distributed between integral ages.

Calculate the probability that a person age 80.5 will die within two years.

(A) 0.0782 (B) 0.0785 (C) 0.0790 (D) 0.0796 (E) 0.0800

7.11. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) $q_x = 0.1$.
(iii) $q_{x+1} = 0.3$.

Calculate $\mathring{e}_{x+0.7:\overline{1}|}$.

7.12. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) $q_{45} = 0.01$.
(iii) $q_{46} = 0.011$.

Calculate $\mathrm{Var}\Big(\min(T_{45}, 2)\Big)$.

7.13. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) ${}_{10}p_x = 0.2$.

Calculate $\mathring{e}_{x:\overline{10}|} - e_{x:\overline{10}|}$.

Exercises continue on the next page . . .

7.14. **[4-F86:21]** You are given:

(i) $q_{60} = 0.020$
(ii) $q_{61} = 0.022$
(iii) Deaths are uniformly distributed over each year of age.

Calculate $\mathring{e}_{60:\overline{1.5}|}$.

(A) 1.447 (B) 1.457 (C) 1.467 (D) 1.477 (E) 1.487

7.15. **[150-F89:21]** You are given:

(i) $q_{70} = 0.040$
(ii) $q_{71} = 0.044$
(iii) Deaths are uniformly distributed over each year of age.

Calculate $\mathring{e}_{70:\overline{1.5}|}$.

(A) 1.435 (B) 1.445 (C) 1.455 (D) 1.465 (E) 1.475

7.16. **[3-S01:33]** For a 4-year college, you are given the following probabilities for dropout from all causes:

$$q_0 = 0.15$$
$$q_1 = 0.10$$
$$q_2 = 0.05$$
$$q_3 = 0.01$$

Dropouts are uniformly distributed over each year.

Compute the temporary 1.5-year complete expected college lifetime of a student entering the second year, $\mathring{e}_{1:\overline{1.5}|}$.

(A) 1.25 (B) 1.30 (C) 1.35 (D) 1.40 (E) 1.45

7.17. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) $\mathring{e}_{x+0.5:\overline{0.5}|} = 5/12$.

Calculate q_x.

7.18. You are given:

(i) Deaths are uniformly distributed over each year of age.
(ii) $\mathring{e}_{55.2:\overline{0.4}|} = 0.396$.

Calculate $\mu_{55.2}$.

7.19. **[150-S87:21]** You are given:

(i) $d_x = k$ for $x = 0, 1, 2, \ldots, \omega - 1$
(ii) $\mathring{e}_{20:\overline{20}|} = 18$
(iii) Deaths are uniformly distributed over each year of age.

Calculate ${}_{30|10}q_{30}$.

(A) 0.111 (B) 0.125 (C) 0.143 (D) 0.167 (E) 0.200

Exercises continue on the next page . . .

7.20. **[150-S89:24]** You are given:

(i) Deaths are uniformly distributed over each year of age.

(ii) $\mu_{45.5} = 0.5$

Calculate $\mathring{e}_{45:\overline{1}|}$.

(A) 0.4 (B) 0.5 (C) 0.6 (D) 0.7 (E) 0.8

7.21. **[CAS3-S04:10]** 4,000 people age (30) each pay an amount, P, into a fund. Immediately after the 1,000th death, the fund will be dissolved and each of the survivors will be paid $50,000.

- Mortality follows the Illustrative Life Table, using linear interpolation at fractional ages.
- $i = 12\%$

Calculate P.

(A) Less than 515
(B) At least 515, but less than 525
(C) At least 525, but less than 535
(D) At least 535, but less than 545
(E) At least 545

Constant force of mortality

7.22. **[160-F87:5]** Based on given values of l_x and l_{x+1}, ${}_{1/4}p_{x+1/4} = 49/50$ under the assumption of constant force of mortality.

Calculate ${}_{1/4}p_{x+1/4}$ under the uniform distribution of deaths hypothesis.

(A) 0.9799 (B) 0.9800 (C) 0.9801 (D) 0.9802 (E) 0.9803

7.23. **[160-S89:5]** A mortality study is conducted for the age interval $(x, x+1]$.

If a constant force of mortality applies over the interval, ${}_{0.25}q_{x+0.1} = 0.05$.

Calculate ${}_{0.25}q_{x+0.1}$ assuming a uniform distribution of deaths applies over the interval.

(A) 0.044 (B) 0.047 (C) 0.050 (D) 0.053 (E) 0.056

7.24. **[150-F89:29]** You are given that $q_x = 0.25$.

Based on the constant force of mortality assumption, the force of mortality is μ^A_{x+s}, $0 < s < 1$.

Based on the uniform distribution of deaths assumption, the force of mortality is μ^B_{x+s}, $0 < s < 1$.

Calculate the smallest s such that $\mu^B_{x+s} \geq \mu^A_{x+s}$.

(A) 0.4523 (B) 0.4758 (C) 0.5001 (D) 0.5239 (E) 0.5477

Exercises continue on the next page . . .

7.25. **[160-S91:4]** From a population mortality study, you are given:

(i) Within each age interval, $[x+k, x+k+1)$, the force of mortality, μ_{x+k}, is constant.

(ii)

k	$e^{-\mu_{x+k}}$	$\dfrac{1-e^{-\mu_{x+k}}}{\mu_{x+k}}$
0	0.98	0.99
1	0.96	0.98

Calculate $\mathring{e}_{x:\overline{2}|}$, the expected lifetime in years over $(x, x+2]$.

(A) 1.92 (B) 1.94 (C) 1.95 (D) 1.96 (E) 1.97

7.26. You are given:

(i) $q_{80} = 0.1$
(ii) $q_{81} = 0.2$
(iii) The force of mortality is constant between integral ages.

Calculate $\mathring{e}_{80.4:\overline{0.8}|}$.

7.27. **[3-S01:27]** An actuary is modeling the mortality of a group of 1000 people, each age 95, for the next three years.

The actuary starts by calculating the expected number of survivors at each integral age by

$$l_{95+k} = 1000\,{}_kp_{95}, \qquad k = 1, 2, 3$$

The actuary subsequently calculates the expected number of survivors at the middle of each year using the assumption that deaths are uniformly distributed over each year of age.

This is the result of the actuary's model:

Age	Survivors
95	1000
95.5	800
96	600
96.5	480
97	—
97.5	288
98	—

The actuary decides to change his assumption for mortality at fractional ages to the constant force assumption. He retains his original assumption for each ${}_kp_{95}$.

Calculate the revised expected number of survivors at age 97.5.

(A) 270 (B) 273 (C) 276 (D) 279 (E) 282

Exercises continue on the next page . . .

7.28. [**M-F06:16**] You are given the following information on participants entering a 2-year program for treatment of a disease:

(i) Only 10% survive to the end of the second year.
(ii) The force of mortality is constant within each year.
(iii) The force of mortality for year 2 is three times the force of mortality for year 1.

Calculate the probability that a participant who survives to the end of month 3 dies by the end of month 21.

(A) 0.61 (B) 0.66 (C) 0.71 (D) 0.75 (E) 0.82

7.29. [**Sample Question #267**] You are given:

(i) $\mu_x = \sqrt{\dfrac{1}{80-x}}, \quad 0 \le x \le 80$
(ii) F is the exact value of $S_0(10.5)$.
(iii) G is the value of $S_0(10.5)$ using the constant force assumption for interpolation between ages 10 and 11.

Calculate $F - G$.

(A) −0.01083 (B) −0.00005 (C) 0 (D) 0.00003 (E) 0.00172

Additional old SOA Exam MLC questions: S12:2, F13:25, F16:1

Additional old CAS Exam 3/3L questions: S05:31, F05:13, S06:13, F06:13, S07:24, S08:16, S09:3, F09:3, S10:4, F10:3, S11:3, S12:3, F12:3, S13:3, F13:3

Additional old CAS Exam LC questions: S14:4, F14:4, S15:3, F15:3

Solutions

7.1. In the second summand, ${}_{1/2}p_x\,\mu_{x+1/2}$ is the density function, which is the constant q_x under UDD. The first summand ${}_{3/4}p_x = 1 - \frac{3}{4}q_x$. So the sum is $1 - \frac{1}{4}q_x$, or $\boxed{{}_{1/4}p_x}$. **(E)**

7.2. Using equation (7.3),

$$\begin{aligned}\frac{3}{31} &= {}_{0.25}q_{x+0.75} = \frac{0.25q_x}{1-0.75q_x}\\ \frac{3}{31} - \frac{2.25}{31}q_x &= 0.25q_x\\ \frac{3}{31} &= \frac{10}{31}q_x\\ q_x &= \boxed{0.3}\end{aligned}$$

7.3. We calculate the probability that $(x+\frac{3}{4})$ survives for half a year. Since the duration crosses an integer boundary, we break the period up into two quarters of a year. The probability of $(x+3/4)$ surviving for 0.25 years is, by equation (7.3),

$${}_{1/4}p_{x+3/4} = \frac{1-0.10}{1-0.75(0.10)} = \frac{0.9}{0.925}$$

The probability of $(x+1)$ surviving to $x+1.25$ is

$$ {}_{1/4}p_{x+1} = 1 - 0.25(0.15) = 0.9625 $$

The answer to the question is then the complement of the product of these two numbers:

$$ {}_{1/2}q_{x+3/4} = 1 - {}_{1/2}p_{x+3/4} = 1 - {}_{1/4}p_{x+3/4}\,{}_{1/4}p_{x+1} = 1 - \left(\frac{0.9}{0.925}\right)(0.9625) = \boxed{\mathbf{0.06351}} $$

Alternatively, you could build a life table starting at age x, with $l_x = 1$. Then $l_{x+1} = (1-0.1) = 0.9$ and $l_{x+2} = 0.9(1-0.15) = 0.765$. Under UDD, l_x at fractional ages is obtained by linear interpolation, so

$$ \begin{aligned} l_{x+0.75} &= 0.75(0.9) + 0.25(1) = 0.925 \\ l_{x+1.25} &= 0.25(0.765) + 0.75(0.9) = 0.86625 \\ {}_{1/2}p_{3/4} &= \frac{l_{x+1.25}}{l_{x+0.75}} = \frac{0.86625}{0.925} = 0.93649 \\ {}_{1/2}q_{3/4} &= 1 - {}_{1/2}p_{3/4} = 1 - 0.93649 = \boxed{\mathbf{0.06351}} \end{aligned} $$

7.4. ${}_{0.3|0.5}q_{x+0.4}$ is ${}_{0.3}p_{x+0.4} - {}_{0.8}p_{x+0.4}$. The first summand is

$$ {}_{0.3}p_{x+0.4} = \frac{1-0.7q_x}{1-0.4q_x} = \frac{1-0.07}{1-0.04} = \frac{93}{96} $$

The probability that $(x+0.4)$ survives to $x+1$ is, by equation (7.3),

$$ {}_{0.6}p_{x+0.4} = \frac{1-0.10}{1-0.04} = \frac{90}{96} $$

and the probability $(x+1)$ survives to $x+1.2$ is

$$ {}_{0.2}p_{x+1} = 1 - 0.2q_{x+1} = 1 - 0.2(0.15) = 0.97 $$

So

$$ {}_{0.3|0.5}q_{x+0.4} = \frac{93}{96} - \left(\frac{90}{96}\right)(0.97) = \boxed{\mathbf{0.059375}} $$

Alternatively, you could use the life table from the solution to the last question, and linearly interpolate:

$$ \begin{aligned} l_{x+0.4} &= 0.4(0.9) + 0.6(1) = 0.96 \\ l_{x+0.7} &= 0.7(0.9) + 0.3(1) = 0.93 \\ l_{x+1.2} &= 0.2(0.765) + 0.8(0.9) = 0.873 \\ {}_{0.3|0.5}q_{x+0.4} &= \frac{0.93 - 0.873}{0.96} = \boxed{\mathbf{0.059375}} \end{aligned} $$

7.5. Under uniform distribution of deaths between integral ages, $l_{x+0.5} = \frac{1}{2}(l_x + l_{x+1})$, since the survival function is a straight line between two integral ages. Therefore, $l_{45.5} = \frac{1}{2}(9{,}164{,}051 + 9{,}127{,}426) = 9{,}145{,}738.5$. Median future lifetime occurs when $l_x = \frac{1}{2}(9{,}145{,}738.5) = 4{,}572{,}869$. This happens between ages 77 and 78. We interpolate between the ages to get the exact median:

$$ \begin{aligned} l_{77} - s(l_{77} - l_{78}) &= 4{,}572{,}869 \\ 4{,}828{,}182 - s(4{,}828{,}182 - 4{,}530{,}360) &= 4{,}572{,}869 \\ 4{,}828{,}182 - 297{,}822s &= 4{,}572{,}869 \\ s = \frac{4{,}828{,}182 - 4{,}572{,}869}{297{,}822} &= \frac{255{,}313}{297{,}822} = 0.8573 \end{aligned} $$

So the median age at death is 77.8573, and median future lifetime is $77.8573 - 45.5 = \boxed{\mathbf{32.3573}}$.

7.6. ${}_xp_0 = \frac{l_x}{l_0} = 1 - \left(\frac{x}{100}\right)^2$. The force of mortality is calculated as the negative derivative of $\ln {}_xp_0$:

$$\mu_x = -\frac{\mathrm{d}\ln {}_xp_0}{\mathrm{d}x} = \frac{2\left(\frac{x}{100}\right)\left(\frac{1}{100}\right)}{1-\left(\frac{x}{100}\right)^2} = \frac{2x}{100^2 - x^2}$$

$$\mu_{50.25} = \frac{100.5}{100^2 - 50.25^2} = 0.0134449$$

For UDD, we need to calculate q_{50}.

$$p_{50} = \frac{l_{51}}{l_{50}} = \frac{1-0.51^2}{1-0.50^2} = 0.986533$$

$$q_{50} = 1 - 0.986533 = 0.013467$$

so under UDD,

$$\mu^L_{50.25} = \frac{q_{50}}{1-0.25q_{50}} = \frac{0.013467}{1-0.25(0.013467)} = 0.013512.$$

The difference between $\mu_{50.25}$ and $\mu^L_{50.25}$ is $0.013445 - 0.013512 =$ **−0.000067**. **(B)**

7.7. $S_0(20) = 1/1.2^4$ and $S_0(21) = 1/1.21^4$, so $q_{20} = 1 - (1.2/1.21)^4 = 0.03265$. Then

$$_{0.4}q_{20.2} = \frac{0.4q_{20}}{1-0.2q_{20}} = \frac{0.4(0.03265)}{1-0.2(0.03265)} = \mathbf{0.01315}$$

7.8.

I. Calculate $_{1|2}q_{36}$.

$$_{1|2}q_{36} = \frac{_2d_{37}}{l_{36}} = \frac{96-87}{99} = 0.09091 \quad ✓$$

This statement does not require uniform distribution of deaths.

II. By equation (7.5),

$$\mu_{37.5} = \frac{q_{37}}{1-0.5q_{37}} = \frac{4/96}{1-2/96} = \frac{4}{94} = 0.042553 \quad ✓$$

III. Calculate $_{0.33}q_{38.5}$.

$$_{0.33}q_{38.5} = \frac{_{0.33}d_{38.5}}{l_{38.5}} = \frac{(0.33)(5)}{89.5} = 0.018436 \quad \times$$

I can't figure out what mistake you'd have to make to get 0.021. **(A)**

7.9. First calculate q_x.

$$1 - 0.75q_x = 0.25$$

$$q_x = 1$$

Then by equation (7.3), $_{0.25}q_{x+0.5} = 0.25/(1-0.5) = 0.5$, making I true.
By equation (7.2), $_{0.5}q_x = 0.5q_x = 0.5$, making II true.
By equation (7.5), $\mu_{x+0.5} = 1/(1-0.5) = 2$, making III false. **(A)**

7.10. We use equation (7.5) to back out q_x for each age.

$$\mu_{x+0.5} = \frac{q_x}{1-0.5q_x} \Rightarrow q_x = \frac{\mu_{x+0.5}}{1+0.5\mu_{x+0.5}}$$
$$q_{80} = \frac{0.0202}{1.0101} = 0.02$$
$$q_{81} = \frac{0.0408}{1.0204} = 0.04$$
$$q_{82} = \frac{0.0619}{1.03095} = 0.06$$

Then by equation (7.3), ${}_{0.5}p_{80.5} = 0.98/0.99$. $p_{81} = 0.96$, and ${}_{0.5}p_{82} = 1 - 0.5(0.06) = 0.97$. Therefore

$$ {}_2q_{80.5} = 1 - \left(\frac{0.98}{0.99}\right)(0.96)(0.97) = \boxed{\mathbf{0.0782}} \qquad \textbf{(A)}$$

7.11. To do this algebraically, we split the group into those who die within 0.3 years, those who die between 0.3 and 1 years, and those who survive one year. Under UDD, those who die will die at the midpoint of the interval (assuming the interval doesn't cross an integral age), so we have

Group	Survival time	Probability of group	Average survival time
I	$(0, 0.3]$	$1 - {}_{0.3}p_{x+0.7}$	0.15
II	$(0.3, 1]$	${}_{0.3}p_{x+0.7} - {}_1p_{x+0.7}$	0.65
III	$(1, \infty)$	${}_1p_{x+0.7}$	1

We calculate the required probabilities.

$${}_{0.3}p_{x+0.7} = \frac{0.9}{0.93} = 0.967742$$
$${}_1p_{x+0.7} = \frac{0.9}{0.93}\bigl(1 - 0.7(0.3)\bigr) = 0.764516$$
$$1 - {}_{0.3}p_{x+0.7} = 1 - 0.967742 = 0.032258$$
$${}_{0.3}p_{x+0.7} - {}_1p_{x+0.7} = 0.967742 - 0.764516 = 0.203226$$
$$\mathring{e}_{x+0.7:\overline{1|}} = 0.032258(0.15) + 0.203226(0.65) + 0.764516(1) = \boxed{\mathbf{0.901452}}$$

Alternatively, we can use trapezoids. We already know from the above solution that the heights of the first trapezoid are 1 and 0.967742, and the heights of the second trapezoid are 0.967742 and 0.764516. So the sum of the area of the two trapezoids is

$$\begin{aligned}\mathring{e}_{x+0.7:\overline{1|}} &= (0.3)(0.5)(1 + 0.967742) + (0.7)(0.5)(0.967742 + 0.764516) \\ &= 0.295161 + 0.606290 = \boxed{\mathbf{0.901451}}\end{aligned}$$

7.12. For the expected value, we'll use the recursive formula. (The trapezoidal rule could also be used.)

$$\begin{aligned}\mathring{e}_{45:\overline{2|}} &= \mathring{e}_{45:\overline{1|}} + p_{45}\,\mathring{e}_{46:\overline{1|}} \\ &= (1 - 0.005) + 0.99(1 - 0.0055) \\ &= 1.979555\end{aligned}$$

We'll use equation (5.7)to calculate the second moment.

$$\begin{aligned}
\mathbf{E}[\min(T_{45},2)^2] &= 2\int_0^2 t\,{}_tp_x\,\mathrm{d}t\\
&= 2\left(\int_0^1 t(1-0.01t)\,\mathrm{d}t + \int_1^2 t(0.99)\big(1-0.011(t-1)\big)\,\mathrm{d}t\right)\\
&= 2\left(\frac{1}{2}-0.01\left(\frac{1}{3}\right)+0.99\left(\frac{(1.011)(2^2-1^2)}{2}-0.011\left(\frac{2^3-1^3}{3}\right)\right)\right)\\
&= 2(0.496667+1.475925) = 3.94518
\end{aligned}$$

So the variance is $3.94518 - 1.979555^2 = \boxed{0.02654}$.

7.13. As discussed on page 132, by equation (7.7), the difference is

$$\frac{1}{2}\,{}_{10}q_x = \frac{1}{2}(1-0.2) = \boxed{0.4}$$

7.14. Those who die in the first year survive $\frac{1}{2}$ year on the average and those who die in the first half of the second year survive 1.25 years on the average, so we have

$$\begin{aligned}
p_{60} &= 0.98\\
{}_{1.5}p_{60} &= 0.98\big(1-0.5(0.022)\big) = 0.96922\\
\mathring{e}_{60:\overline{1.5|}} &= 0.5(0.02)+1.25(0.98-0.96922)+1.5(0.96922) = \boxed{1.477305} \qquad \textbf{(D)}
\end{aligned}$$

Alternatively, we use the trapezoidal method. The first trapezoid has heights 1 and $p_{60} = 0.98$ and width 1. The second trapezoid has heights $p_{60} = 0.98$ and ${}_{1.5}p_{60} = 0.96922$ and width 1/2.

$$\begin{aligned}
\mathring{e}_{60:\overline{1.5|}} &= \frac{1}{2}(1+0.98)+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(0.98+0.96922)\\
&= \boxed{1.477305} \qquad \textbf{(D)}
\end{aligned}$$

7.15. $p_{70} = 1-0.040 = 0.96$, ${}_2p_{70} = (0.96)(0.956) = 0.91776$, and by linear interpolation, ${}_{1.5}p_{70} = 0.5(0.96+0.91776) = 0.93888$. Those who die in the first year survive 0.5 years on the average and those who die in the first half of the second year survive 1.25 years on the average. So

$$\mathring{e}_{70:\overline{1.5|}} = 0.5(0.04)+1.25(0.96-0.93888)+1.5(0.93888) = \boxed{1.45472} \qquad \textbf{(C)}$$

Alternatively, we can use the trapezoidal method. The first year's trapezoid has heights 1 and 0.96 and width 1 and the second year's trapezoid has heights 0.96 and 0.93888 and width 1/2, so

$$\mathring{e}_{70:\overline{1.5|}} = 0.5(1+0.96)+0.5(0.5)(0.96+0.93888) = \boxed{1.45472} \qquad \textbf{(C)}$$

7.16. First we calculate ${}_tp_1$ for $t = 1, 2$.

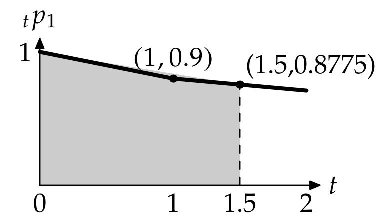
$$\begin{aligned}
p_1 &= 1-q_1 = 0.90\\
{}_2p_1 &= (1-q_1)(1-q_2) = (0.90)(0.95) = 0.855
\end{aligned}$$

By linear interpolation, ${}_{1.5}p_1 = (0.5)(0.9+0.855) = 0.8775$.

The algebraic method splits the students into three groups: first year dropouts, second year (up to time 1.5) dropouts, and survivors. In each dropout group survival on the average is to the midpoint (0.5 years for the first group, 1.25 years for the second group) and survivors survive 1.5 years. Therefore

$$\mathring{e}_{1:\overline{1.5|}} = 0.10(0.5) + (0.90 - 0.8775)(1.25) + 0.8775(1.5) = \boxed{1.394375} \qquad \textbf{(D)}$$

Alternatively, we could sum the two trapezoids making up the shaded area at the right.

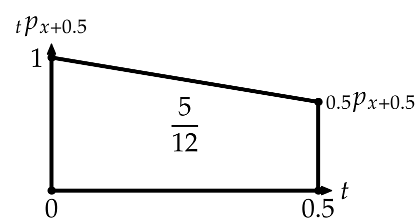


$$\begin{aligned}\mathring{e}_{1:\overline{1.5|}} &= (1)(0.5)(1+0.9) + (0.5)(0.5)(0.90+0.8775)\\ &= 0.95 + 0.444375 = \boxed{1.394375} \qquad \textbf{(D)}\end{aligned}$$

7.17. Those who die survive 0.25 years on the average and survivors survive 0.5 years, so we have

$$\begin{aligned}0.25\,{}_{0.5}q_{x+0.5} + 0.5\,{}_{0.5}p_{x+0.5} &= \tfrac{5}{12}\\ 0.25\left(\frac{0.5q_x}{1-0.5q_x}\right) + 0.5\left(\frac{1-q_x}{1-0.5q_x}\right) &= \tfrac{5}{12}\\ 0.125q_x + 0.5 - 0.5q_x &= \tfrac{5}{12} - \tfrac{5}{24}q_x\\ \frac{1}{2} - \frac{5}{12} &= \left(-\frac{5}{24} + \frac{1}{2} - \frac{1}{8}\right)q_x\\ \frac{1}{12} &= \frac{q_x}{6}\\ q_x &= \boxed{\tfrac{1}{2}}\end{aligned}$$

Alternatively, complete life expectancy is the area of the trapezoid shown on the right, so



$$\frac{5}{12} = 0.5(0.5)(1 + {}_{0.5}p_{x+0.5})$$

Then ${}_{0.5}p_{x+0.5} = \frac{2}{3}$, from which it follows

$$\begin{aligned}\frac{2}{3} &= \frac{1-q_x}{1-\frac{1}{2}q_x}\\ q_x &= \boxed{\tfrac{1}{2}}\end{aligned}$$

7.18. Survivors live 0.4 years and those who die live 0.2 years on the average, so

$$0.396 = 0.4_{0.4}p_{55.2} + 0.2_{0.4}q_{55.2}$$

Using the formula ${}_{0.4}q_{55.2} = 0.4q_{55}/(1-0.2q_{55})$ (equation (7.3)), we have

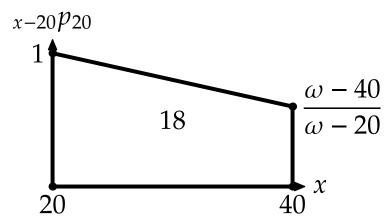
$$\begin{aligned}0.4\left(\frac{1-0.6q_{55}}{1-0.2q_{55}}\right) + 0.2\left(\frac{0.4q_{55}}{1-0.2q_{55}}\right) &= 0.396\\ 0.4 - 0.24q_{55} + 0.08q_{55} &= 0.396 - 0.0792q_{55}\\ 0.0808q_{55} &= 0.004\end{aligned}$$

$$q_{55} = \frac{0.004}{0.0808} = 0.0495$$
$$\mu_{55.2} = \frac{q_{55}}{1-0.2q_{55}} = \frac{0.0495}{1-0.2(0.0495)} = \boxed{\mathbf{0.05}}$$

7.19. Since d_x is constant for all x and deaths are uniformly distributed within each year of age, mortality is uniform globally. We back out ω using equation (5.12), $\mathring{e}_{x:\overline{n}|} = {}_np_x(n) + {}_nq_x(n/2)$:

$$\begin{aligned}
10\,{}_{20}q_{20} + 20\,{}_{20}p_{20} &= 18\\
10\left(\frac{20}{\omega-20}\right) + 20\left(\frac{\omega-40}{\omega-20}\right) &= 18\\
200 + 20\omega - 800 &= 18\omega - 360\\
2\omega &= 240\\
\omega &= 120
\end{aligned}$$

Alternatively, we can back out ω using the trapezoidal rule. Complete life expectancy is the area of the trapezoid shown to the right.



$$\begin{aligned}
\mathring{e}_{20:\overline{20}|} = 18 &= (20)(0.5)\left(1+\frac{\omega-40}{\omega-20}\right)\\
0.8 &= \frac{\omega-40}{\omega-20}\\
0.8\omega - 16 &= \omega - 40\\
0.2\omega &= 24\\
\omega &= 120
\end{aligned}$$

Once we have ω, we compute

$${}_{30|10}q_{30} = \frac{10}{\omega-30} = \frac{10}{90} = \boxed{\mathbf{0.1111}} \qquad \textbf{(A)}$$

7.20. We use equation (7.5) to obtain

$$\begin{aligned}
0.5 &= \frac{q_x}{1-0.5q_x}\\
q_x &= 0.4
\end{aligned}$$

Then $\mathring{e}_{45:\overline{1}|} = 0.5\big(1 + (1-0.4)\big) = \boxed{\mathbf{0.8}}$. **(E)**

7.21. According to the Illustrative Life Table, $l_{30} = 9{,}501{,}381$, so we are looking for the age x such that $l_x = 0.75(9{,}501{,}381) = 7{,}126{,}036$. This is between 67 and 68. Using linear interpolation, since $l_{67} = 7{,}201{,}635$ and $l_{68} = 7{,}018{,}432$, we have

$$x = 67 + \frac{7{,}201{,}635 - 7{,}126{,}036}{7{,}201{,}635 - 7{,}018{,}432} = 67.4127$$

This is 37.4127 years into the future. $\frac{3}{4}$ of the people collect 50,000. We need $50{,}000\left(\frac{3}{4}\right)\left(\frac{1}{1.12^{37.4127}}\right) = \boxed{\mathbf{540.32}}$ per person. **(D)**

7.22. Under constant force, ${}_sp_{x+t} = p_x^s$, so $p_x = {}_{1/4}p_{x+1/4}^4 = 0.98^4 = 0.922368$ and $q_x = 1 - 0.922368 = 0.077632$. Under uniform distribution of deaths,

$$\begin{aligned}{}_{1/4}p_{x+1/4} &= 1 - \frac{(1/4)q_x}{1-(1/4)q_x}\\ &= 1 - \frac{(1/4)(0.077632)}{1-(1/4)(0.077632)}\\ &= 1 - 0.019792 = \boxed{0.980208} \qquad \textbf{(D)}\end{aligned}$$

7.23. Under constant force, ${}_sp_{x+t} = p_x^s$, so $p_x = 0.95^4 = 0.814506$, $q_x = 1 - 0.814506 = 0.185494$. Then under a uniform assumption,

$${}_{0.25}q_{x+0.1} = \frac{0.25q_x}{1-0.1q_x} = \frac{(0.25)(0.185494)}{1-0.1(0.185494)} = \boxed{0.047250} \qquad \textbf{(B)}$$

7.24. Using constant force, μ^A is a constant equal to $-\ln p_x = -\ln 0.75 = 0.287682$. Then

$$\begin{aligned}\mu^B_{x+s} &= \frac{q_x}{1-sq_x} = 0.287682\\ \frac{0.25}{1-0.25s} &= 0.287682\\ 0.2877 - 0.25(0.287682)s &= 0.25\\ s &= \frac{0.287682-0.25}{(0.25)(0.287682)} = \boxed{0.5239} \qquad \textbf{(D)}\end{aligned}$$

7.25. We integrate ${}_tp_x$ from 0 to 2. Between 0 and 1, ${}_tp_x = e^{-t\mu_x}$.

$$\int_0^1 e^{-t\mu_x}\,dt = \frac{1-e^{-\mu_x}}{\mu_x} = 0.99$$

Between 1 and 2, ${}_tp_x = p_x\ {}_{t-1}p_{x+1} = 0.98e^{-(t-1)\mu_{x+1}}$.

$$\int_1^2 e^{-(t-1)\mu_{x+1}}\,dt = \frac{1-e^{-\mu_{x+1}}}{\mu_{x+1}} = 0.98$$

So the answer is $0.99 + 0.98(0.98) = \boxed{1.9504}$. **(C)**

7.26.

$$\begin{aligned}\mathring{e}_{80.4:\overline{0.8}|} &= \mathring{e}_{80.4:\overline{0.6}|} + {}_{0.6}p_{80.4}\,\mathring{e}_{81:\overline{0.2}|}\\ &= \frac{\int_{0.4}^1 0.9^t\,dt}{0.9^{0.4}} + 0.9^{0.6}\int_0^{0.2} 0.8^t\,dt\\ &= \frac{0.9^{0.6}-1}{\ln 0.9} + \left(0.9^{0.6}\right)\frac{0.8^{0.2}-1}{\ln 0.8}\\ &= 0.581429 + (0.938740)(0.195603) = \boxed{0.765049}\end{aligned}$$

7.27. Under uniform distribution, the numbers of deaths in each half of the year are equal, so if 120 deaths occurred in the first half of $x = 96$, then 120 occurred in the second half, and $l_{97} = 480 - 120 = 360$. Then if ${}_{0.5}q_{97} = (360 - 288)/360 = 0.2$, then $q_{97} = 2\,{}_{0.5}q_{97} = 0.4$, so $p_{97} = 0.6$. Under constant force, ${}_{1/2}p_{97} = p_{97}^{0.5} = \sqrt{0.6}$. The answer is $360\sqrt{0.6} =$ **278.8548**. **(D)**

7.28. Let μ be the force of mortality in year 1. Then 10% survivorship means

$$\begin{aligned} e^{-\mu-3\mu} &= 0.1 \\ e^{-4\mu} &= 0.1 \end{aligned}$$

The probability of survival 21 months given survival 3 months is the probability of survival 9 months after month 3, or $e^{-(3/4)\mu}$, times the probability of survival another 9 months given survival 1 year, or $e^{-(3/4)3\mu}$, which multiplies to $e^{-3\mu} = (e^{-4\mu})^{3/4} = 0.1^{3/4} = 0.177828$, so the death probability is $1 - 0.177828 =$ **0.822172**. **(E)**

7.29. The exact value is:

$$F = {}_{10.5}p_0 = \exp\left(-\int_0^{10.5} \mu_x \,\mathrm{d}x\right)$$

$$\begin{aligned} \int_0^{10.5} (80-x)^{-0.5}\mathrm{d}x &= -2(80-x)^{0.5}\Big|_0^{10.5} \\ &= -2\left(69.5^{0.5} - 80^{0.5}\right) = 1.215212 \\ {}_{10.5}p_0 &= e^{-1.215212} = 0.296647 \end{aligned}$$

To calculate $S_0(10.5)$ with constant force interpolation between 10 and 11, we have ${}_{0.5}p_{10} = p_{10}^{0.5}$, and ${}_{10.5}p_0 = {}_{10}p_0\,{}_{0.5}p_{10}$, so

$$\begin{aligned} \int_0^{10} (80-x)^{-0.5}\mathrm{d}x &= -2\left(70^{0.5} - 80^{0.5}\right) = 1.155343 \\ \int_{10}^{11} (80-x)^{-0.5}\mathrm{d}x &= -2\left(69^{0.5} - 70^{0.5}\right) = 0.119953 \\ G = {}_{10.5}p_0 &= e^{-1.155343-0.5(0.119953)} = 0.296615 \end{aligned}$$

Then $F - G = 0.296647 - 0.296615 =$ **0.000032**. **(D)**

Quiz Solutions

7-1. Notice that $\mu_{50.4} = \frac{q_{50}}{1-0.4q_{50}}$ while ${}_{0.6}q_{50.4} = \frac{0.6q_{50}}{1-0.4q_{50}}$, so ${}_{0.6}q_{50.4} = 0.6(0.01) =$ **0.006**

7-2. The algebraic method goes: those who die will survive 0.3 on the average, and those who survive will survive 0.6.

$$\begin{aligned} {}_{0.6}q_{x+0.4} &= \frac{0.6(0.1)}{1-0.4(0.1)} = \frac{6}{96} \\ {}_{0.6}p_{x+0.4} &= 1 - \frac{6}{96} = \frac{90}{96} \\ \mathring{e}_{x+0.4:\overline{0.6}|} &= \frac{6}{96}(0.3) + \frac{90}{96}(0.6) = \frac{55.8}{96} = \mathbf{0.58125} \end{aligned}$$

The geometric method goes: we need the area of a trapezoid having height 1 at $x + 0.4$ and height 90/96 at $x+1$, where 90/96 is ${}_{0.6}p_{x+0.4}$, as calculated above. The width of the trapezoid is 0.6. The answer is therefore $0.5\,(1 + 90/96)\,(0.6) = \boxed{\mathbf{0.58125}}$.

7-3. Batteries failing in month 1 survive an average of 0.5 month, those failing in month 2 survive an average of 1.5 months, and those failing in month 3 survive an average of 2.125 months (the average of 2 and 2.25). By linear interpolation, ${}_{2.25}q_0 = 0.25(0.6) + 0.75(0.2) = 0.3$. So we have

$$\begin{aligned}\mathring{e}_{0:\overline{2.25}|} &= q_0(0.5) + {}_{1|}q_0(1.5) + {}_{2|0.25}q_0(2.125) + {}_{2.25}p_0(2.25)\\ &= (0.05)(0.5) + (0.20 - 0.05)(1.5) + (0.3 - 0.2)(2.125) + 0.70(2.25) = \boxed{\mathbf{2.0375}}\end{aligned}$$

Lesson 8

Survival Distributions: Select Mortality

Reading: *Actuarial Mathematics for Life Contingent Risks* 2nd edition 3.6–3.10

Suppose you selected two 60-year-old men from the population. The first one was selected randomly, whereas the second one had recently purchased a life insurance policy. Would the mortality rate for both of these, q_{60} be the same? No. The second person was underwritten for a life insurance policy, which means his medical situation was reviewed. He had to satisfy certain guidelines regarding weight, blood pressure, blood lipids, family history, existing medical conditions, and possibly even driving record and credit history. The fact he was approved for an insurance policy implies that his mortality rate is lower than that of a randomly selected 60-year-old male.

Not only would q_{60} be different. If both men survived 5 years, q_{65} would be different as well. A man whose health was established 5 years ago will have better mortality than a randomly selected man.

A mortality table for the insured population must consider both the age of issue and the duration since issue. Mortality rates would require two arguments and need a notation like $q(x,t)$ where x is the issue age and t the duration since issue.

International Actuarial Notation provides two-parameter notation for all actuarial functions. The parameters are written as subscripts with a bracket around the first parameter and a plus sign between the parameters. In other words, the subscript is of the form $[x]+t$. When $t=0$, it is omitted. Thus the mortality rate for a 60-year-old who just purchased a life insurance policy would be written as $q_{[60]}$. The mortality rate for a 65-year-old who purchased a policy at age 60 would be written $q_{[60]+5}$. When mortality depends on the initial age as well as duration, it is known as *select* mortality, since the person is *selected* at that age.

Working with select mortality is no different from working with non-select mortality (sometimes known as *aggregate* mortality), as long as the bracketed parameter is not changed. For example:

Example 8A You are given:

(i) $l_{[45]}=1000$

(ii) ${}_5q_{[45]}=0.04$

(iii) ${}_5q_{[45]+5}=0.05$

Calculate $l_{[45]+10}$.

Answer:

$$l_{[45]+10}=l_{[45]}\,{}_5p_{[45]}\,{}_5p_{[45]+5}=(1000)(0.96)(0.95)=\boxed{912}$$ □

Example 8B You are given that $q_{[60]+k}=0.02$ for $k=0,1,2,\ldots$

Calculate $e_{[60]}$.

Answer: First of all,

$$_kp_{[60]}=\prod_{j=1}^{k}\left(1-q_{[60]+j-1}\right)=0.98^k$$

Then by equation (5.18),

$$e_{[60]}=\sum_{k=1}^{\infty}{}_kp_{[60]}=\sum_{k=1}^{\infty}0.98^k=\frac{0.98}{0.02}=\boxed{49}$$ □

In these examples, we see that everything we learned until now for (x) applies to $[x]$ as well.

Maintaining a full two dimensional table with all possible ages and durations may be overkill. After many years, the lower mortality resulting from selection wears off. A 60-year-old who purchased insurance 40 years ago may be no healthier on the average than a randomly selected 60-year-old. Therefore, typically it is assumed that after a certain number of years, selection has no effect: $q_{[x]+t} = q_{x+t}$ if t is at least as high as the select period. The mortality after the select period is called *ultimate* mortality.

EXAMPLE 8C Select mortality rates for [45] are half of the Illustrative Life Table's mortality rates for a selection period of 3 years.

Calculate ${}_{2|2}q_{[45]}$.

ANSWER: We will calculate ${}_2p_{[45]}$ and ${}_2p_{[45]+2}$. For $q_{[45]}$ and $q_{[45]+1}$, we use half of the ILT rates.

$${}_2p_{[45]} = \big(1 - 0.5(0.004)\big)\big(1 - 0.5(0.00431)\big) = 0.995849$$

$q_{[45]+2} = 0.5q_{47}$, but $q_{[45]+3} = q_{48}$ since the selection period ends after 3 years. Mortality for duration 3 and on is no different from standard mortality.

$${}_2p_{[45]+2} = \big(1 - 0.5(0.00466)\big)(1 - 0.00504) = 0.992642$$

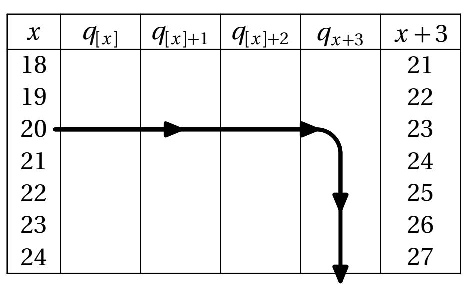
The answer is ${}_{2|2}q_{[45]} = {}_2p_{[45]}(1 - {}_2p_{[45]+2}) = 0.995849(1 - 0.992642) =$ **0.007327**. □

Quiz 8-1 Select mortality rates for [45] are half of the Illustrative Life Table's mortality rates for a selection period of five years.

Calculate ${}_{1|}q_{[45]}$.

Selection, especially at older ages, does not truly wear off for a long time. Mortality tables used by insurance companies typically have 25-year select periods, and even that may not be enough at older ages. However, tables used in exams must be short, and typically have selection periods of 3 years or less.

A select-and-ultimate mortality table is shown in tabular form by listing the issue ages vertically and the durations horizontally. If the select period is n years, there are n columns for n durations, followed by a column with ultimate mortality. The columns are arranged so that to find the mortality at all durations for a specific issue age, you read across the row corresponding to that issue age and when you hit the end of the row, you continue from that cell down the last column. Computing mortality for issue age 20 is shown schematically in the following figure:



Suppose you were given Table 8.1 as a select and ultimate mortality table. Let's compute ${}_2q_{[40]+2}$. This is $q_{[40]+2} + p_{[40]+2}\, q_{43}$, because $q_{[40]+3} = q_{43}$ when the select period is 3 years. The mortality rate $q_{[40]+2}$ is read off horizontally on the row for age 40, and is the entry 0.008. The mortality rate q_{43} is the next entry horizontally, or 0.012. The answer is $0.008 + 0.992(0.012) = 0.019904$.

Table 8.1: Select and ultimate mortality table

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
40	0.002	0.005	0.008	0.012	43
41	0.003	0.006	0.009	0.015	44
42	0.004	0.007	0.010	0.018	45

Quiz 8-2 Mortality is select and ultimate and is shown in Table 8.1.
Calculate ${}_{2|3}q_{[41]}$.

Notice that there is no direct way to go from $[x]$ to $[x+1]$. A person who is selected at age x will be $[x]$ in the first year, $[x]+1$ in the second year, and so on, until he becomes ultimate at age $x+k$, and then his age will continue to increase and be without brackets. He will never be selected again. He will never be $[x+1]$.

A life selected at age x can never become a life selected at any higher age. $[x]$ will never become $[x+1]$.

If you need to relate $[x]$ and $[x+1]$, the only way to do it is to go through the ultimate table.

Example 8D In a 1-year select and ultimate table, you are given

(i) $e_{[60]} = 25$
(ii) $q_{60} = 0.01$
(iii) $q_{61} = 0.02$
(iv) $q_{[x]} = 0.6q_x$

Calculate $e_{[61]}$.

Answer: We need to calculate e_{61}, then e_{62}, and then finally we can calculate $e_{[61]}$. Each calculation will use the life expectancy recursive formula, equation (6.4).

$$\begin{aligned} e_{61} &= \frac{e_{[60]}}{p_{[60]}} - 1 = \frac{25}{1-0.6(0.01)} - 1 = 24.15091 \\ e_{62} &= \frac{e_{61}}{p_{61}} - 1 = \frac{24.15091}{1-0.02} - 1 = 23.64378 \\ e_{[61]} &= p_{[61]}(1+e_{62}) = \big(1-0.6(0.02)\big)(1+23.64378) = \boxed{24.34805} \end{aligned}$$

□

Suppose we wanted to construct a select-and-ultimate life table. We would want to start with a radix like 10,000,000 in the upper leftmost cell. But we would also want the tables to merge in the rightmost column. To build the table, we would have to start with the first age and duration—let's say age 0 duration 0—then calculate $l_{[0]+t}$ through the end of the select period n, then l_x for $x \ge n$. After that, we would have to work backwards to fill in the select table for all starting ages other than 0. For example, using l_4, we would compute $l_{[1]+2} = l_4/p_{[1]+2}$, then $l_{[1]+1} = l_{[1]+2}/p_{[1]+1}$, then $l_{[1]} = l_{[1]+1}/p_{[1]}$. The next example illustrates this procedure using the mortality rates of Table 8.1.

Example 8E You are given:

(i) Mortality rates are select and ultimate with a select period of 3 years, and are given in Table 8.1.

(ii) $l_{[40]} = 10{,}000{,}000$.

Compute $l_{[42]}$.

Answer: We must compute l_{45} recursively.

$$\begin{aligned}
l_{[40]+1} &= 10{,}000{,}000(1-0.002) = 9{,}980{,}000\\
l_{[40]+2} &= 9{,}980{,}000(1-0.005) = 9{,}930{,}100\\
l_{43} &= 9{,}930{,}100(1-0.008) = 9{,}850{,}659\\
l_{44} &= 9{,}850{,}659(1-0.012) = 9{,}732{,}451\\
l_{45} &= 9{,}732{,}451(1-0.015) = 9{,}586{,}464
\end{aligned}$$

So far, the life table looks like this:

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	l_{x+3}	$x+3$
40	10,000,000	9,980,000	9,930,100	9,850,659	43
41				9,732,451	44
42				9,586,464	45

Now we work backwards from l_{45} to $l_{[42]}$.

$$\begin{aligned}
l_{[42]+2} &= \frac{9{,}586{,}464}{1-0.010} = 9{,}683{,}297\\
l_{[42]+1} &= \frac{9{,}683{,}297}{1-0.007} = 9{,}751{,}558\\
l_{[42]} &= \frac{9{,}751{,}558}{1-0.004} = \boxed{9{,}790{,}721}
\end{aligned}$$

The life table now looks like this:

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	l_{x+3}	$x+3$
40	10,000,000	9,980,000	9,930,100	9,850,659	43
41				9,732,451	44
42	9,790,721	9,751,558	9,683,297	9,586,464	45

For additional practice, fill in the row for [41]. The completed table is in the footnote.[1] □

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[1]

$$\begin{aligned}
l_{[41]+2} &= \frac{9{,}732{,}451}{1-0.009} = 9{,}820{,}839\\
l_{[41]+1} &= \frac{9{,}820{,}839}{1-0.006} = 9{,}880{,}120\\
l_{[41]} &= \frac{9{,}880{,}120}{1-0.003} = 9{,}909{,}850
\end{aligned}$$

The complete table looks like this:

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	l_{x+3}	$x+3$
40	10,000,000	9,980,000	9,930,100	9,850,659	43
41	9,909,850	9,880,120	9,820,839	9,732,451	44
42	9,790,721	9,751,558	9,683,297	9,586,464	45

Exam questions may ask you to fill in blanks in a life table, or to use select-and-ultimate mortality as you would use aggregate mortality to compute mortality at fractional ages or life expectations. One frequent question is going from $l_{[x]}$ to $l_{[x+1]}$. We did something similar in the last example. The road is crooked: first you have to go from $l_{[x]}$ to l_{x+n+1}, where n is the length of the select period, and then work backwards to $l_{[x+1]}$.

Quiz 8-3 For a 2-year select-and-ultimate table, you are given

(i) $q_{80} = 0.1$
(ii) $q_{81} = 0.2$
(iii) $q_{82} = 0.3$
(iv) $l_{[80]} = 1000$
(v) $q_{[x]} = 0.5q_x$
(vi) $q_{[x]+1} = 0.8q_{x+1}$

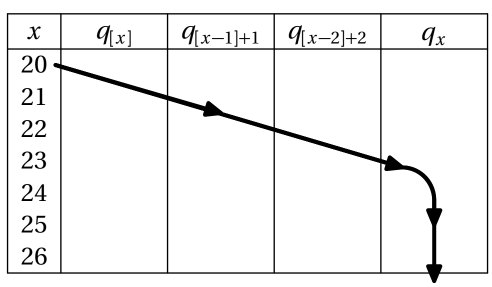
Calculate $l_{[81]}$.

So far, all of our select-and-ultimate tables had rows corresponding to issue ages. An alternative arrangement of the table would be to have one row for each *attained* age.[2] Here's what a version of Table 8.1 would look like with this arrangement:

Table 8.2: Select and ultimate mortality table

x	$q_{[x]}$	$q_{[x-1]+1}$	$q_{[x-2]+2}$	q_x
40	0.002	0.004	0.006	0.008
41	0.003	0.005	0.007	0.009
42	0.004	0.006	0.008	0.011
43	0.005	0.007	0.009	0.012

Some values needed to be added to this table, and indeed it would be impossible to have a table like this starting at an age less than the select period. You can identify the structure of the table from the column headings. To use a table like this to compute ${}_kp_{[x]}$, you need to go down a diagonal until you reach the ultimate column, and then to go vertically down. The schematic is



EXAMPLE 8F Using Table 8.2, calculate ${}_3p_{[40]}$.

ANSWER:

$$ {}_3p_{[40]} = (1-0.002)(1-0.005)(1-0.008) = \boxed{0.985066} $$ □

[2] I have never seen a select-and-ultimate table arranged this way in real life, but the textbook features this arrangement, so perhaps it is common in the UK.

Exercises

8.1. You are given the following extract from a 2-year select and ultimate mortality table:

x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}
80	1000	950	900
81	—	920	—
82	—	—	860

You are given that $q_{[80]+1} = q_{[81]+1}$.

Calculate $_{1|}q_{[80]+1}$.

8.2. You are given the following information from a 2-year select and ultimate mortality table:

(i)

x	q_x
90	0.10
91	0.12
92	0.13
93	0.15
94	0.16

(ii) $q_{[x]} = 0.5q_x$

(iii) $q_{[x]+1} = 0.75q_{x+1}$

(iv) $l_{[91]} = 10{,}000$

Calculate $l_{[90]}$.

8.3. For a select-and-ultimate mortality table you are given:

(i) Ultimate mortality is uniformly distributed with limiting age $\omega = 120$.

(ii) During the 2-year select period, $\mu_{[x]+t} = (\mu_{x+t})(t/2)$

Calculate $e_{[64]}$.

8.4. In a 2-year select and ultimate mortality table, deaths are uniformly distributed between integral ages and durations. You are given the following mortality rates:

t	$_{t\|}q_{[30]}$	$_{t\|}q_{[31]}$
0	0.05	0.05
1	0.06	0.07
2	0.08	
3	0.08	

Calculate $_{2}q_{[31]+0.5}$.

Exercises continue on the next page . . .

8.5. **[150-F87:2]** You are given the following extract from a 3-year select and ultimate mortality table:

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	l_{x+3}	$x+3$
70	—	—	—	7600	73
71	—	7984	—	—	74
72	8016	—	7592	—	75

Assume:

(i) Mortality in the ultimate table is uniformly distributed.
(ii) $d_{[x]} = d_{[x]+1} = d_{[x]+2}$, $x = 70, 71, 72$, where $d_{[x]+t} = l_{[x]+t} - l_{[x]+t+1}$.

Calculate $1000({}_{2|2}q_{[71]})$.

(A) 26.73 (B) 32.43 (C) 43.37 (D) 47.83 (E) 48.99

8.6. **[150-S90:21]** For a 2-year select and ultimate mortality table, you are given:

(i) Ultimate mortality follows the Illustrative Life Table.
(ii) $q_{[x]} = 0.5q_x$ for all x.
(iii) $q_{[x]+1} = 0.5q_{x+1}$ for all x.
(iv) $l_{[96]} = 10{,}000$.

Calculate $l_{[97]}$.

8.7. **[150-S91:10]** For a two-year select-and-ultimate mortality table, you are given:

(i) $q_{[x]} = (1-2k)q_x$
(ii) $q_{[x]+1} = (1-k)q_{x+1}$
(iii) $l_{[32]} = 90$
(iv) $l_{32} = 100$
(v) $l_{33} = 90$
(vi) $l_{34} = 63$

Calculate $l_{[32]+1}$.

(A) 82 (B) 83 (C) 84 (D) 85 (E) 86

8.8. **[3-F00:10]** You are given the following extract from a select-and-ultimate mortality table with a 2-year select period:

x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
60	80,625	79,954	78,839	62
61	79,137	78,402	77,252	63
62	77,575	76,770	75,578	64

Assume that deaths are uniformly distributed between integral ages.

Calculate ${}_{0.9}q_{[60]+0.6}$.

(A) 0.0102 (B) 0.0103 (C) 0.0104 (D) 0.0105 (E) 0.0106

Exercises continue on the next page . . .

8.9. **[CAS4-F82:17]** The table below represents a section of a select and ultimate mortality table. It shows the rates of mortality $q_{[x]+n}$ at attained age $x+n$ among a group of lives insured at age x.

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}
21	0.00120	0.00150	0.00170	0.00180
22	0.00125	0.00155	0.00175	0.00185
23	0.00130	0.00160	0.00180	0.00195

You are given that $l_{[21]} = 1{,}000{,}000$.

Calculate $l_{[22]}$.

(A) Less than 997,500
(B) At least 997,500, but less than 998,000
(C) At least 998,000, but less than 998,500
(D) At least 998,500, but less than 999,000
(E) At least 999,000

8.10. **[CAS4-S87:17]** (1 point) You are given the following select-and-ultimate mortality table:

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	$q_{[x]+3}$	q_{x+4}
33	0.02	0.015	0.03	0.025	0.035
34	0.01	0.025	0.02	0.03	0.04
35	0.02	0.015	0.03	0.035	0.05
36	0.01	0.025	0.03	0.045	0.04
37	0.02	0.025	0.04	0.035	0.03
38	0.02	0.035	0.03	0.025	0.035
39	0.03	0.025	0.02	0.035	0.045
40	0.02	0.015	0.03	0.04	0.04
41	0.01	0.025	0.035	0.035	0.035
42	0.02	0.03	0.03	0.03	0.035

Calculate the probability that a life age 36 who has been insured for two years will live to age 40.

(A) Less than 0.86
(B) At least 0.86, but less than 0.87
(C) At least 0.87, but less than 0.88
(D) At least 0.88, but less than 0.89
(E) At least 0.89

8.11. **[150-S98:1]** For a select-and-ultimate mortality table applicable to patients after heart surgery, you are given:

(i) Ultimate mortality follows the Illustrative Life Table.
(ii) During the four-year select period,

$$p_{[x]+k} = (0.80 + 0.05k)p_{x+k}, \qquad k = 0, 1, 2, 3$$

Calculate ${}_4p_{[62]+2}$.

(A) 0.531 (B) 0.539 (C) 0.781 (D) 0.857 (E) 0.916

Exercises continue on the next page . . .

8.12. **[CAS4A-S94:7]** (2 points) You are given the following excerpt from a select-and-ultimate table with a 2-year select period:

$[x]$	$1000q_{[x]}$	$1000q_{[x]+1}$	$1000q_{x+2}$
40	0.438	0.574	0.699
41	0.453	0.599	0.738
42	0.477	0.634	0.790
43	0.510	0.680	0.856
44	0.551	0.737	0.937

Calculate $1000_{1|2}q_{[41]}$.

(A) Less than 1.20
(B) At least 1.20, but less than 1.40
(C) At least 1.40, but less than 1.60
(D) At least 1.60, but less than 1.80
(E) At least 1.80

8.13. **[CAS4A-S97:10]** (2 points) You are given two groups of people. Group 1 consists of 100,000 people, each age 30, selected at age 30. Their mortality is described by the following illustrative select-and-ultimate mortality table:

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}
29	0.00130	0.00134	0.00138	0.00142
30	0.00132	0.00136	0.00140	0.00144
31	0.00134	0.00138	0.00142	0.00146
32	0.00136	0.00140	0.00144	0.00148

Group 2 also contains 100,000 people, each age 30, taken from the general population. Their mortality is described in the following table:

t	$_{t\mid}q_{30}$
0	0.00138
1	0.00140
2	0.00144
3	0.00147

Calculate how many more people from Group 1 survive to age 32 than do people from Group 2.

(A) Less than 7
(B) At least 7, but less than 9
(C) At least 9, but less than 11
(D) At least 11, but less than 13
(E) At least 13

Exercises continue on the next page . . .

8.14. **[CAS4A-F98:14]** (2 points) The following is extracted from a select and ultimate mortality table with a 2-year select period:

$[x]$	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
24	—	—	42,683	26
25	—	—	35,000	27
26	—	—	26,600	28

For all x, $q_{[x]+1} = 1.5q_{[x+1]}$ and $q_{[x]+2} = 1.2q_{[x+1]+1}$.

Determine $l_{[26]}$.

(A) Less than 35,000
(B) At least 35,000, but less than 35,500
(C) At least 35,500, but less than 36,000
(D) At least 36,000, but less than 36,500
(E) At least 36,500

8.15. You are given the following select-and-ultimate table with a 3-year select period:

x	$q_{[x]}$	$q_{[x-1]+1}$	$q_{[x-2]+2}$	q_x
60	0.005	0.007	0.009	0.013
61	0.006	0.008	0.010	0.015
62	0.007	0.009	0.012	0.018
63	0.008	0.011	0.015	0.022

Calculate ${}_3q_{[60]+1}$.

8.16. **[3-F01:2]** For a select-and-ultimate mortality table with a 3-year select period:

(i)

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
60	0.09	0.11	0.13	0.15	63
61	0.10	0.12	0.14	0.16	64
62	0.11	0.13	0.15	0.17	65
63	0.12	0.14	0.16	0.18	66
64	0.13	0.15	0.17	0.19	67

(ii) White was a newly selected life on 01/01/2000.
(iii) White's age on 01/01/2001 is 61.
(iv) P is the probability on 01/01/2001 that White will be alive on 01/01/2006.

Calculate P.

(A) $0 \le P < 0.43$
(B) $0.43 \le P < 0.45$
(C) $0.45 \le P < 0.47$
(D) $0.47 \le P < 0.49$
(E) $0.49 \le P \le 1.00$

Exercises continue on the next page . . .

8.17. You are given the following 2-year select-and-ultimate table:

x	$q_{[x]}$	$q_{[x-1]+1}$	q_x
80	0.015	0.023	0.035
81	0.020	0.030	0.040
82	0.023	0.039	0.055
83	0.027	0.044	0.065

A life table has $l_{[80]} = 955$.

Determine $l_{[81]}$.

8.18. For a 2-year select and ultimate mortality table, you are given

(i) $\mu_{[37]+t} = \mu_{37+t} - A$, $0 \le t \le 2$.
(ii) $\mathring{e}_{[37]} = 58$.
(iii) $\mathring{e}_{[37]:\overline{2}|} = 1.9$.
(iv) $\mathring{e}_{37:\overline{2}|} = 1.7$.
(v) $\mathring{e}_{37} = 57.5$.

Determine A.

8.19. You are given the following 2-year select-and-ultimate mortality table:

x	$l_{[x]}$	$l_{[x-1]+1}$	l_x
65	85,000	86,000	87,000
66	82,000	83,200	84,500
67	79,000	80,200	81,800
68	76,000	77,100	79,000

Deaths are uniformly distributed between integral ages.

Calculate ${}_{0.7|1.1}q_{[65]+0.5}$.

8.20. For a 1-year select and ultimate mortality table, $p_{[x]} = p_x + 0.001$. Deaths are uniformly distributed between integral ages and durations. The complete life expectancy for $(x + 1)$ is 78.

Calculate $\mathring{e}_{[x]} - \mathring{e}_x$.

Use the following information for questions 8.21 and 8.22:

You are given the following select-and-ultimate life table with a two year select period.

x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}
90			
91	1250		920
92	1000	900	

You are also given that $q_{[x]+t} = \dfrac{t+1}{3} q_{x+t}$.

8.21. Calculate $l_{[90]+1}$.

8.22. Calculate l_{94}.

Exercises continue on the next page . . .

Use the following information for questions 8.23 and 8.24:

The force of mortality for a life selected at age x follows the following model:

$$\mu_{[x]+t} = \phi(x)\mu_t, \qquad t \geq 0$$

You are given:

(i) $\phi(x) = \beta_0 + \beta_1 x$

(ii) $\mu_t = t$, $t \geq 0$

(iii) $p_{[0]} = 0.96$

(iv) ${}_3p_{[35]} = 1.25\,{}_3p_{[65]}$

8.23. [150-S98:39] Calculate β_0.

(A) 0.04 (B) 0.05 (C) 0.06 (D) 0.07 (E) 0.08

8.24. [150-S98:40] Calculate β_1.

(A) 0.0012 (B) 0.0017 (C) 0.0025 (D) 0.0034 (E) 0.0050

8.25. [150-S87:12] From a life table with a one-year select period, you are given:

x	$l_{[x]}$	$d_{[x]}$	$\mathring{e}_{[x]}$
85	1000	100	5.556
86	850	100	

Assume that deaths are uniformly distributed over each year of age.

Calculate $\mathring{e}_{[86]}$.

(A) 5.04 (B) 5.13 (C) 5.22 (D) 5.30 (E) 5.39

Use the following information for questions 8.26 and 8.27:

The force of mortality for a life selected at age x follows the model:

$$\mu_{[x]+t} = \phi(x)\mu(t), \qquad t \geq 0$$

You are given:

- $\phi(x) = \beta + 0.006S + 0.003x$
- $\mu(t) = t$
- $S = \begin{cases} 1, & \text{if } (x) \text{ smokes} \\ 0, & \text{otherwise} \end{cases}$
- ${}_{10}p^n_{[30]} = 0.96$
- A superscript of s indicates the case where $S = 1$ and n indicates the case where $S = 0$.

8.26. [C3 Sample:21] Determine x such that $q^s_{[35]} = q^n_{[x]}$.

Exercises continue on the next page . . .

8.26–27. *(Repeated for convenience) Use the following information for questions 8.26 and 8.27:*

> The force of mortality for a life selected at age x follows the model:
>
> $$\mu_{[x]+t} = \phi(x)\mu(t), \qquad t \geq 0$$
>
> You are given:
>
> - $\phi(x) = \beta + 0.006S + 0.003x$
> - $\mu(t) = t$
> - $S = \begin{cases} 1, & \text{if } (x) \text{ smokes} \\ 0, & \text{otherwise} \end{cases}$
> - ${}_{10}p^{n}_{[30]} = 0.96$
> - A superscript of s indicates the case where $S = 1$ and n indicates the case where $S = 0$.

8.27. [**C3 Sample:22**] Calculate the probability that a life, drawn at random from a population of lives selected at age 30 of which 40% are smokers, will survive at least 10 years.

8.28. [**150-83-96:3**] For a ten-year select-and-ultimate mortality table, you are given:

(i) $l_{[30]+t} = \dfrac{\sqrt{60}}{9}\left(1 - \dfrac{t}{100}\right), \quad 0 \leq t < 10$

(ii) $l_{30+t} = \dfrac{\sqrt{70-t}}{10}, \quad 10 \leq t \leq 70$

Calculate $\mathring{e}_{[30]}$.

(A) 21.0 (B) 39.0 (C) 42.0 (D) 45.5 (E) 48.5

8.29. [**M-S05:28**] For a life table with a one-year select period, you are given:

(i)

x	$l_{[x]}$	$d_{[x]}$	l_{x+1}	$\mathring{e}_{[x]}$
80	1000	90	—	8.5
81	920	90	—	—

(ii) Deaths are uniformly distributed over each year of age.

Calculate $\mathring{e}_{[81]}$.

(A) 8.0 (B) 8.1 (C) 8.2 (D) 8.3 (E) 8.4

Exercises continue on the next page . . .

8.30. **[MLC-S07:18]** You are given the following extract from a 2-year select-and-ultimate mortality table:

$[x]$	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
65	—	—	8200	67
66	—	—	8000	68
67	—	—	7700	69

The following relationships hold for all x:

(i) $3q_{[x]+1} = 4q_{[x+1]}$
(ii) $4q_{x+2} = 5q_{[x+1]+1}$

Calculate $l_{[67]}$.

(A) 7940 (B) 8000 (C) 8060 (D) 8130 (E) 8200

8.31. **[150-F89:A6]**(4 points) For a select-and-ultimate mortality table with a one-year select period, you are given:

x	$l_{[x]}$	$d_{[x]} = q_{[x]}l_{[x]}$	$\mathring{e}_{[x]}$
85	1000	100	5.225
86	850	100	

Assume deaths are uniformly distributed over each year of age.

(a) Calculate $p_{[85]}$, $p_{[86]}$, and p_{86}.

(b) Calculate $\int_2^\infty {}_tp_{[85]}\,\mathrm{d}t$.

(c) Calculate $\mathring{e}_{[86]}$.

8.32. In a 5-year select and ultimate mortality table,

(i) The force of mortality during the ultimate period is $\mu_x = A + Bc^x$.
(ii) During the select period, $\mu_{[x]+t} = \mu_{x+t}(0.9^{5-t})$.

Demonstrate that

$$ {}_tp_{[x]} = \exp\left(-\frac{A(0.9^5 - 0.9^{5-t})}{\ln 0.9} - 0.9^{5-t}Bc^x\frac{c^t - 0.9^t}{\ln c - \ln 0.9}\right) \text{ for } 0 \le t \le 5 $$

Additional old SOA Exam MLC questions: S12:1,13, F12:2, S13:19, F13:3, S14:2, F14:20, S17:2

Additional old CAS Exam 3/3L questions: S06:12, F06:10

Solutions

8.1. We need l_{83}, which would be the number in the row for $x = 81$, column l_{x+2}. We can compute $p_{[80]+1}$ using the life table:

$$ p_{[80]+1} = \frac{l_{82}}{l_{[80]+1}} = \frac{900}{950} $$

Since $q_{[81]+1} = q_{[80]+1}$, it follows that $p_{[81]+1} = p_{[80]+1}$. We use this to calculate l_{83}:

$$\begin{aligned} l_{83} &= l_{[81]+1} p_{[81]+1} \\ &= 920\left(\frac{900}{950}\right) = 871.58 \\ {}_{1|}q_{[80]+1} &= \frac{l_{82} - l_{83}}{l_{[80]+1}} \\ &= \frac{900 - 871.58}{950} = \boxed{\mathbf{0.02992}} \end{aligned}$$

8.2. We must advance from $l_{[91]}$ to the end of the select period at l_{93}, go back to l_{92}, and then back to $l_{[90]}$.

$$\begin{aligned} l_{93} &= (10{,}000)(1 - q_{[91]})(1 - q_{[91]+1}) \\ &= (10{,}000)(1 - 0.5q_{91})(1 - 0.75q_{92}) \\ &= (10{,}000)(0.94)(0.9025) = 8483.5 \\ l_{92} &= 8483.5/(1 - q_{92}) \\ &= 8483.5/0.87 = 9751.15 \\ l_{[90]} &= \frac{9751.15}{(1 - q_{[90]+1})(1 - q_{[90]})} \\ &= \frac{9751.15}{(1 - 0.5q_{90})(1 - 0.75q_{91})} \\ &= \frac{9751.15}{(0.91)(0.95)} = \boxed{\mathbf{11{,}279.53}} \end{aligned}$$

8.3. First we calculate e_{66}, using the fact that under uniform mortality, $\mathring{e}_x = (\omega - x)/2$.

$$e_{66} = \mathring{e}_{66} - 0.5 = (120 - 66)/2 - 0.5 = 26.5$$

Then we relate $e_{[64]}$ to e_{66} using recursion twice:

$$\begin{aligned} e_{[64]} &= p_{[64]} + p_{[64]}\, e_{[64]+1} \\ &= p_{[64]} + p_{[64]}\big(p_{[64]+1}(1 + e_{66})\big) \\ &= p_{[64]} + {}_2p_{[64]}(1 + e_{66}) \end{aligned}$$

So we have to calculate $p_{[64]}$ and ${}_2p_{[64]}$ from μ (formula (3.7) on page 38). Let's calculate ${}_tp_{[64]}$ for any $0 < t \le 2$ and then plug in $t = 1, 2$.

$$\begin{aligned} \mu_{64+u} &= \frac{1}{56 - u} && \text{by uniform mortality (formula (4.7) on page 67)} \\ \mu_{[64]+u} &= \frac{u}{2(56 - u)} \end{aligned}$$

$$\begin{aligned}
\int_0^t \mu_{[64]+u}\,du &= \int_0^t \frac{u\,du}{2(56-u)} \\
&= \frac{1}{2}\int_0^t \left(-1+\frac{56}{56-u}\right)du \\
&= \frac{1}{2}\Bigl(-u-56\ln(56-u)\Bigr)\Bigr|_0^t \\
&= \frac{-t+56\bigl(\ln 56-\ln(56-t)\bigr)}{2} \\
{}_tp_{[64]} &= \exp\left(-\int_0^t \mu_{[64]+u}\,du\right) \\
p_{[64]} &= \exp\left(-\frac{-1+56(\ln 56-\ln 55)}{2}\right) = \exp(-0.00451816) = 0.995492 \\
{}_2p_{[64]} &= \exp\left(-\frac{-2+56(\ln 56-\ln 54)}{2}\right) = \exp(-0.0182940) = 0.981872 \\
e_{[64]} &= 0.995492 + 0.981872(1+26.5) = \boxed{27.997}
\end{aligned}$$

As a result of selection, life expectancy is longer than for an ordinary 64-year old, for whom $e_{64} = 27.5$.

8.4. We need q_{33}. We will back it out of the values of ${}_{t|}q_{[30]}$.

$$\begin{aligned}
q_{33} &= \frac{d_{33}}{l_{33}} = \frac{d_{33}/l_{[30]}}{l_{33}/l_{[30]}} = \frac{{}_{3|}q_{[30]}}{{}_3p_{[30]}} \\
{}_3p_{[30]} &= 1-0.05-0.06-0.08 = 0.81 \\
q_{33} &= \frac{0.08}{0.81} = \frac{8}{81}
\end{aligned}$$

Now we can construct a life table just for [31]. Start with $l_{[31]} = 10{,}000$. Then

$$\begin{aligned}
l_{[31]+1} &= 10{,}000(0.95) = 9{,}500 \\
l_{33} &= 9{,}500 - 0.07(10{,}000) = 8{,}800 \\
l_{34} &= 8{,}800\left(1-\frac{8}{81}\right) = 7{,}930.864
\end{aligned}$$

Interpolate to calculate $l_{[31]+0.5} = 0.5(10{,}000 + 9{,}500) = 9{,}750$ and $l_{[31]+2.5} = 0.5(8{,}800 + 7{,}930.864) = 8{,}365.432$. Then

$$ {}_2q_{[31]+0.5} = 1 - \frac{8{,}365.432}{9{,}750} = \boxed{0.142007} $$

8.5. We need $l_{[71]}$, $l_{[71]+2}$, and l_{75}.

Using (ii) on the row of $x = 72$, we have that

$$ d_{[72]} + d_{[72]+1} = l_{[72]} - l_{[72]+2} = 8016 - 7592 = 424, $$

so $d_{[72]} = 212$ and

$$\begin{aligned}
l_{[72]+1} &= 8016 - 212 = 7804 \\
l_{75} &= 7592 - 212 = 7380
\end{aligned}$$

By (i), $d_{73} = d_{74}$, and $d_{73} + d_{74} = 7600 - 7380 = 220$, so

$$d_{73} = 110$$
$$l_{74} = 7600 - 110 = 7490$$

Since $l_{[71]+1} - l_{74} = 7984 - 7490 = 494$,

$$d_{[71]+1} = \frac{494}{2} = 247$$
$$l_{[71]} = 7984 + 247 = 8231$$
$$l_{[71]+2} = 7984 - 247 = 7737$$

Finally,

$$1000({}_{2|2}q_{[71]}) = 1000\left(\frac{l_{[71]+2} - l_{75}}{l_{[71]}}\right)$$
$$= 1000\left(\frac{7737 - 7380}{8231}\right) = \boxed{43.3726} \qquad \textbf{(C)}$$

8.6. We will need to advance from [96] to 99 and then back to [97]. In other words,

$$l_{[97]} = l_{[96]}\frac{{}_3p_{[96]}}{{}_2p_{[97]}}$$

From the Illustrative Life Table, $q_{96} = 0.30445$ and $q_{97} = 0.32834$. Then

$$l_{[96]+1} = 10{,}000\big(1 - (0.5)(0.30445)\big) = 8477.75$$
$$l_{[96]+2} = l_{98} = 8477.75\big(1 - (0.5)(0.3283410)\big) = 7085.95$$

In the Illustrative Life Table, $q_{98} = 0.35360$, so

$$l_{99} = 7085.95(1 - 0.35360) = 4580.36$$
$$l_{[97]} = \frac{l_{99}}{{}_2p_{[97]}}$$
$$= \frac{4580.36}{\big(1 - 0.5(0.35360)\big)\big(1 - 0.5(0.32834)\big)} = \boxed{6656.97}$$

The original answer choices were based on a different Illustrative Life Table.

8.7. From the given information, $p_{32} = 0.9$, $p_{33} = 0.7$, and ${}_2p_{[32]} = 0.7$. To solve for k:

$$0.7 = {}_2p_{[32]} = p_{[32]}p_{[32]+1}$$
$$= \big(1 - (1 - 2k)(0.1)\big)\big(1 - (1 - k)(0.3)\big)$$
$$= (0.9 + 0.2k)(0.7 + 0.3k)$$
$$= 0.06k^2 + 0.41k + 0.63$$

$$0.06k^2 + 0.41k - 0.07 = 0$$
$$k = \frac{1}{6}$$
$$q_{[32]} = (1 - 2k)q_{32} = \left(1 - 2(\tfrac{1}{6})\right)(0.1) = \tfrac{1}{15}$$
$$l_{[32]+1} = 90\left(1 - \tfrac{1}{15}\right) = \boxed{84} \qquad \textbf{(C)}$$

8.8. It is easiest to work this out by calculating $l_{[60]+0.6}$ and $l_{[60]+1.5}$ using interpolation.

$$l_{[60]+0.6} = 80{,}625 - 0.6(80{,}625 - 79{,}954) = 80{,}222.4$$
$$l_{[60]+1.5} = 79{,}954 - 0.5(79{,}954 - 78{,}839) = 79{,}396.5$$
$${}_{0.9}q_{[60]+0.6} = 1 - \frac{79{,}396.5}{80{,}222.4} = \boxed{0.01030} \qquad \textbf{(B)}$$

8.9. As usual, advance to l_{25} and then back to $l_{[22]}$.

$$l_{25} = 1{,}000{,}000(1-0.00120)(1-0.00150)(1-0.00170)(1-0.00180) = 993{,}814.30$$
$$l_{[22]} = \frac{993{,}814.30}{(1-0.00175)(1-0.00155)(1-0.00125)} = \boxed{998{,}350} \qquad \textbf{(C)}$$

8.10. Since the life is age 36 and has been insured for two years, it is $[34]+2$. The probability of surviving to age 40 is

$$\begin{aligned} p_{[34]+2}\, p_{[34]+3}\, p_{38}\, p_{39} &= (1-0.02)(1-0.03)(1-0.04)(1-0.05) \\ &= (0.98)(0.97)(0.96)(0.95) = \boxed{0.8669472} \qquad \textbf{(B)} \end{aligned}$$

The select-and-ultimate table is reproduced below with the probabilities we used in gray.

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	$q_{[x]+3}$	q_{x+4}
33	0.02	0.015	0.03	0.025	0.035
34	0.01	0.025	0.02	0.03	0.04
35	0.02	0.015	0.03	0.035	0.05
36	0.01	0.025	0.03	0.045	0.04
37	0.02	0.025	0.04	0.035	0.03
38	0.02	0.035	0.03	0.025	0.035
39	0.03	0.025	0.02	0.035	0.045
40	0.02	0.015	0.03	0.04	0.04
41	0.01	0.025	0.035	0.035	0.035
42	0.02	0.03	0.03	0.03	0.035

8.11. Let $s_k = p_{[x]+k}/p_{x+k}$ be the select factor, the ratio of select to ultimate mortality for the same age. The select factors are $s_2 = 0.90$ and $s_3 = 0.95$ in this exercise. Therefore:

$$\begin{aligned} {}_4p_{[62]+2} &= p_{[62]+2}p_{[62]+3}p_{66}p_{67} \\ &= (p_{64}\, s_2)(p_{65}\, s_3)p_{66}p_{67} = (0.90)(0.95)\,{}_4p_{64} \end{aligned}$$

From the Illustrative Life Table,

$${}_4p_{64} = \frac{l_{68}}{l_{64}} = \frac{7{,}018{,}432}{7{,}683{,}979} = 0.913385$$

so the answer is

$${}_4p_{[62]+2} = (0.9)(0.95)(0.913385) = \boxed{0.78094} \qquad \textbf{(C)}$$

8.12.

$$\begin{aligned} 1000\,{}_{1|2}q_{[41]} &= 1000p_{[41]}\;{}_2q_{[41]+1} \\ &= 1000(1-0.000453)\big(0.000599 + (1-0.000599)(0.000738)\big) \\ &= \boxed{1.3360} \qquad \textbf{(B)} \end{aligned}$$

8.13. In group 1, survival is

$$100{,}000(1-0.00132)(1-0.00136) = 99{,}732.18$$

In group 2, survival is

$$100{,}000(1-0.00138-0.00140) = 99{,}722$$

The difference is $\boxed{\mathbf{10.18}}$ **(C)**

8.14. We need to go back from l_{28} to $l_{[26]}$, so we'll need $q_{[26]+1}$ and $q_{[26]}$.

$$\begin{aligned}
q_{[26]+1} &= \frac{q_{[25]+2}}{1.2} = \frac{(35{,}000-26{,}600)/35{,}000}{1.2} = 0.2\\
q_{[26]} &= \frac{q_{[25]+1}}{1.5} = \frac{q_{[24]+2}}{(1.5)(1.2)}\\
&= \frac{(42{,}683-35{,}000)/42{,}683}{1.8} = 0.1\\
l_{[26]} &= \frac{26{,}600}{(1-0.2)(1-0.1)} = \boxed{\mathbf{36{,}944}} \qquad \textbf{(E)}
\end{aligned}$$

8.15.

$$\begin{aligned}
{}_3p_{[60]+1} &= p_{[60]+1}\,p_{[60]+2}\,p_{63}\\
&= (1-0.008)(1-0.012)(1-0.022) = 0.958534\\
{}_3q_{[60]+1} &= 1-0.958534 = \boxed{\mathbf{0.041466}}
\end{aligned}$$

8.16. Since White was 61 on 01/01/2001, he was 60 when selected a year earlier, so we use the $x = 60$ row of the table, starting 1 year after selection, or starting at $q_{[60]+1} = 0.11$. The complements of the 5 mortality rates to use reading across to the end of the row and then down are:

$$\begin{aligned}
{}_5p_{[60]+1} &= (1-0.11)(1-0.13)(1-0.15)(1-0.16)(1-0.17)\\
&= (0.89)(0.87)(0.85)(0.84)(0.83) = \boxed{\mathbf{0.458866}} \qquad \textbf{(C)}
\end{aligned}$$

8.17. We need to calculate $l_{82} = l_{80}\,p_{[80]}\,p_{[80]+1}$, then $l_{83} = l_{82}\,p_{82}$, and then use $l_{83} = l_{[81]}\,p_{[81]}\,p_{[81]+1}$.

$$\begin{aligned}
l_{82} &= 955(1-0.015)(1-0.030) = 912.45\\
l_{83} &= 912.45(1-0.055) = 862.27\\
l_{[81]} &= \frac{862.27}{(1-0.039)(1-0.020)} = \boxed{\mathbf{915.57}}
\end{aligned}$$

8.18. The recursive formula, equation (6.1), relates $\mathring{e}_{37}$, $\mathring{e}_{37:\overline{2}|}$, and ${}_2p_{37}$. We will use it to obtain a relationship between ${}_2p_{37}$ and ${}_2p_{[37]}$. For select mortality:

$$\begin{aligned}
\mathring{e}_{[37]} &= \mathring{e}_{[37]:\overline{2}|} + {}_2p_{[37]}\,\mathring{e}_{39}\\
58 &= 1.9 + {}_2p_{[37]}\,\mathring{e}_{39}\\
56.1 &= {}_2p_{[37]}\,\mathring{e}_{39}
\end{aligned}$$

For ultimate mortality:

$$\begin{aligned}\mathring{e}_{37} &= \mathring{e}_{37:\overline{2}|} + {}_2p_{37}\,\mathring{e}_{39}\\ 57.5 &= 1.7 + {}_2p_{37}\,\mathring{e}_{39}\\ 55.8 &= {}_2p_{37}\,\mathring{e}_{39}\end{aligned}$$

We therefore can relate the select and ultimate survival probabilities:

$$\frac{{}_2p_{[37]}}{{}_2p_{37}} = \frac{56.1}{55.8} = 1.005376$$

As indicated at the bottom of Table 3.1, adding a constant k to μ multiplies the survival probability by e^{-kt}, so

$$\begin{aligned}e^{2A} &= 1.005376\\ A &= \frac{\ln 1.005376}{2} = \boxed{\mathbf{0.002681}}\end{aligned}$$

8.19.

$${}_{0.7|1.1}q_{[65]+0.5} = \frac{l_{[65]+1.2} - l_{67.3}}{l_{[65]+0.5}}$$

We compute the three l's that we need using linear interpolation.

$$\begin{aligned}l_{[65]+0.5} &= 0.5(l_{[65]} + l_{[65]+1}) = 0.5(85{,}000 + 83{,}200) = 84{,}100\\ l_{[65]+1.2} &= 0.8l_{[65]+1} + 0.2l_{67} = 0.8(83{,}200) + 0.2(81{,}800) = 82{,}920\\ l_{67.3} &= 0.7l_{67} + 0.3l_{68} = 0.7(81{,}800) + 0.3(79{,}000) = 80{,}960\\ {}_{0.7|1.1}q_{[65]+0.5} &= \frac{82{,}920 - 80{,}960}{84{,}100} = \boxed{\mathbf{0.023306}}\end{aligned}$$

8.20. Under uniform distribution of deaths $\mathring{e}_x - e_x = \frac{1}{2}$, so $e_{x+1} = 77.5$. By the recursive formula for curtate expectation, $e_x = p_x(1 + e_{x+1})$, and this is true for $[x]$ as well as for (x). So

$$e_{[x]} - e_x = (p_{[x]} - p_x)(1 + e_{x+1}) = 0.001(1 + 77.5) = 0.0785$$

But since the difference between $\mathring{e}_x$ and e_x is always $\frac{1}{2}$, $\mathring{e}_{[x]} - \mathring{e}_x$ is also $\boxed{\mathbf{0.0785}}$.

8.21. We need to go from l_{93} to l_{92} and from there to $l_{[90]+1}$. To get q_{92} , we use $q_{[92]} = q_{92}/3$:

$$\begin{aligned}q_{[92]} &= 1 - \frac{900}{1000} = 0.1\\ q_{92} &= 3q_{[92]} = 0.3\\ l_{92} &= \frac{920}{1 - 0.3} = \frac{920}{0.7}\end{aligned}$$

To get $q_{[90]+1}$, we use $q_{[90]+1} = (2/3)q_{91}$, and $q_{[91]} = q_{91}/3$, so $q_{[90]+1} = 2q_{[91]}$. We also know that $q_{[91]+1} = (2/3)q_{92} = 0.2$ and from the life table, ${}_2p_{[91]} = 920/1250$. Putting this information together,

$$\begin{aligned}\frac{920}{1250} &= (1 - q_{[91]})(1 - 0.2)\\ q_{[91]} &= 1 - \frac{920/1250}{0.8} = 0.08 = \frac{q_{91}}{3}\end{aligned}$$

$$q_{[90]+1} = \frac{2q_{91}}{3} = 2q_{[91]} = 2(0.08) = 0.16$$

Finally,

$$l_{[90]+1} = \frac{920}{0.7(1-0.16)} = \boxed{1564.63}$$

8.22. The trick is to relate $l_{93} = 920$ to $l_{[92]+1} = 900$, knowing that they have to merge at the next duration and knowing the relationship between the select and ultimate mortality rates.

From the life table, we have $q_{[92]+1} = 1 - l_{94}/900$ and $q_{93} = 1 - l_{94}/920$, but $q_{[92]+1} = \frac{2}{3}q_{93}$, so

$$\begin{aligned}
1 - \frac{l_{94}}{900} &= \frac{2}{3}\left(1 - \frac{l_{94}}{920}\right)\\
\frac{l_{94}}{900} &= \frac{1}{3} + \frac{2}{3}\left(\frac{l_{94}}{920}\right)\\
\frac{3l_{94}}{900} &= 1 + \frac{2l_{94}}{920}\\
l_{94}\left(\frac{3}{900} - \frac{2}{920}\right) &= 1\\
l_{94} &= \frac{1}{\frac{3}{900} - \frac{2}{920}} = \boxed{862.5}
\end{aligned}$$

8.23. We will use $p_x = \exp\left(-\int_0^1 \mu_{x+s}\,\mathrm{d}s\right)$.

$$\begin{aligned}
\mu_{[0]+t} &= \phi(0)\mu_t = \beta_0 t\\
-\ln p_{[0]} = -\ln 0.96 &= \int_0^1 \mu_{[0]+t}\,\mathrm{d}t = \int_0^1 \beta_0 t\,\mathrm{d}t\\
0.040822 &= \frac{\beta_0}{2}\\
\beta_0 &= \boxed{0.081644} \qquad \textbf{(E)}
\end{aligned}$$

8.24. From (i) and (ii),

$$\mu_{[x]+t} = (\beta_0 + \beta_1 x)t$$

We use (iv) now, in conjunction with ${}_3p_x = \exp\left(-\int_0^3 \mu_{x+s}\,\mathrm{d}s\right)$. Logging and negating this expression,

$$-\ln {}_3p_{[35]} = \int_0^3 (\beta_0 + 35\beta_1)t\,\mathrm{d}t$$

But (iv) says ${}_3p_{[35]} = 1.25\,{}_3p_{[65]}$, so

$$-\ln {}_3p_{[35]} = -\ln 1.25 - \ln {}_3p_{[65]}$$

and $-\ln {}_3p_{[65]} = \int_0^3 (\beta_0 + 65\beta_1)t\,\mathrm{d}t$. Therefore

$$\int_0^3 (\beta_0 + 35\beta_1)t\,\mathrm{d}t = -\ln 1.25 + \int_0^3 (\beta_0 + 65\beta_1)t\,\mathrm{d}t$$

$$\int_0^3 t\,\mathrm{d}t = \frac{t^2}{2}\bigg|_0^3 = 4.5$$
$$(4.5)(\beta_0) + (4.5)(35\beta_1) = -\ln 1.25 + (4.5)(\beta_0) + (4.5)(65\beta_1)$$
$$(4.5)(30)\beta_1 = \ln 1.25$$
$$\beta_1 = \frac{\ln 1.25}{135} = \frac{0.223144}{135} = \boxed{0.001653} \qquad \textbf{(B)}$$

8.25. Using the recursive formula for curtate expectation, we need to go from $e_{[85]}$ to e_{87} and then back to $e_{[86]}$. In order to do that, we'll need l_{86} and l_{87}. From the life table, $l_{86} = l_{[85]} - d_{[85]} = 900$ and $l_{87} = l_{[86]} - d_{[86]} = 750$. With $p_{[85]} = 1 - d_{[85]}/l_{[85]} = 0.9$ and $p_{86} = 1 - d_{86}/l_{86} = \frac{5}{6}$ we are ready to use the recursive formula for curtate expectations.

$e_{[85]} = 5.556 - 0.5 = 5.056$ due to uniform distribution of deaths. By the recursive formula for curtate expectation

$$5.056 = 0.9(1 + e_{86}) \quad \Rightarrow e_{86} = 4.617778$$
$$4.617778 = \frac{5}{6}(1 + e_{87}) \quad \Rightarrow e_{87} = 4.54133$$
$$e_{[86]} = \left(1 - \frac{100}{850}\right)(1 + 4.54133) = 4.8894$$
$$\mathring{e}_{[86]} = 4.8894 + 0.5 = \boxed{5.3894} \qquad \textbf{(E)}$$

8.26. Since the only dependence on x is in $\phi(x)$, we only have to arrange that $\phi^s(35) = \beta + 0.006 + 0.003(35)$ equals $\phi^n(x) = \beta + 0.003x$. This means $0.003x = 0.006 + 0.003(35)$. Thus $x = \boxed{37}$.

8.27. The survival distribution of the population is a mixture with 40% weight on smoker survival and 60% weight on non-smoker survival, so the probability that a randomly drawn member survives 10 years is

$${}_{10}p_{[30]} = 0.6\,{}_{10}p^n_{[30]} + 0.4\,{}_{10}p^s_{[30]}$$

We are given that ${}_{10}p^n_{[30]} = 0.96$. For smokers, $\phi^s(x)$ differs from $\phi^n(x)$ only by $0.006S = 0.006$, so

$$\mu^s_{x+t} = \phi^s(t)\mu_t = \big(\phi^n(t) + 0.006\big)\mu_t = \mu^n_{x+t} + 0.006t$$

Since ${}_tp_x = \exp\left(-\int_0^t \mu_{[x]+u}\,\mathrm{d}u\right)$, adding $0.006t$ to μ_{x+t} results in multiplying the survival probability ${}_tp_x$ by the factor $\exp\left(-\int_0^t 0.006u\,\mathrm{d}u\right)$. This multiplicative factor is

$$\exp\left(-\int_0^{10} 0.006u\,\mathrm{d}u\right) = \exp\left(-0.003u^2\right)\Big|_0^{10} = e^{-0.3} = 0.740818.$$

Therefore, the probability that a randomly drawn life will survive 10 years is

$$0.96\big(0.6(1) + 0.4(0.740818)\big) = \boxed{0.8605}$$

8.28. We can write $\mathring{e}_{[30]}$ as

$$\mathring{e}_{[30]} = \int_0^{10} {}_tp_{[30]}\,\mathrm{d}t + {}_{10}p_{[30]}\,\mathring{e}_{40} \qquad (*)$$

Select mortality is linear with annual rate ${}_{k|}q_{[30]} = 0.01$. So ${}_tp_{[30]} = (100 - t)/100$, and the integral in (*) is the value of the integrand at the middle of the integration interval ($t = 5$) times 10.

$$\int_0^{10} {}_tp_{[30]}\,\mathrm{d}t = 10\,{}_5p_{[30]} = 10(0.95) = 9.5$$

Ultimate mortality has a beta distribution with $\alpha = 0.5$ and $\omega = 100$, and therefore has mean

$$\mathring{e}_{40} = \left(\frac{\omega - x}{\alpha + 1}\right) = \frac{100 - 40}{1/2 + 1} = 40$$

Also ${}_{10}p_{[30]} = 1 - 10/100 = 0.9$.

Plugging these three values into (*),

$$\mathring{e}_{[30]} = 9.5 + 0.9(40) = \boxed{45.5} \qquad \textbf{(D)}$$

8.29. The official solution gives several methods.

First of all, fill in l_{x+1}: $l_{81} = 1000 - 90 = 910$ and $l_{82} = 920 - 90 = 830$.

The most straightforward method is to use the recursion

$$\mathring{e}_x = \int_0^1 {}_tp_x\,\mathrm{d}t + p_x\,\mathring{e}_{x+1}$$

and for UDD, $\int_0^1 {}_tp_x\,\mathrm{d}t = (1 + p_x)/2$, the average value of ${}_tp_x$. So

$$\mathring{e}_x = \frac{1 + p_x}{2} + p_x\,\mathring{e}_{x+1}$$
$$\mathring{e}_{x+1} = \frac{\mathring{e}_x - (1 + p_x)/2}{p_x}$$

We do two recursions to go from $\mathring{e}_{[80]}$ to $\mathring{e}_{81}$ to $\mathring{e}_{82}$, and then one recursion back to $\mathring{e}_{[81]}$:

$$p_{[80]} = \frac{1000 - 90}{1000} = 0.91$$
$$p_{81} = \frac{l_{82}}{l_{81}} = \frac{920 - 90}{1000 - 90} = \frac{83}{91}$$
$$p_{[81]} = \frac{920 - 90}{920} = \frac{83}{92}$$
$$\mathring{e}_{81} = \frac{8.5 - 1.91/2}{0.91} = 8.2912$$
$$\mathring{e}_{82} = \frac{8.2912 - (1 + 83/91)/2}{83/91} = 8.0422$$
$$\mathring{e}_{[81]} = 8.0422(83/92) + \frac{1 + 83/92}{2} = \boxed{8.2065} \qquad \textbf{(C)}$$

Alternatively, with UDD, $e_x = \mathring{e}_x - 0.5$, so one can work with curtate expectation and the recursion $e_x = p_x(1 + e_{x+1})$:

$$e_{81} = \frac{8}{0.91} - 1 = 7.7912$$

$$\begin{aligned}
e_{82} &= \frac{7.7912}{83/91} - 1 = 7.5422 \\
e_{[81]} &= (1 + 7.5422)(83/92) = 7.7065 \\
\mathring{e}_{[81]} &= 7.7065 + 0.5 = \boxed{8.2065}
\end{aligned}$$

It is unusual that select mortality is higher than ultimate, but perhaps it could happen for an annuity.

8.30. To get $q_{[67]+1}$, we use (ii), since that has a left side that only depends on ultimate mortality. $q_{[67]+1} = 0.8q_{68}$, and $q_{68} = 1 - 7700/8000 = 0.0375$, so $q_{[67]+1} = 0.03$.

To get $q_{[67]}$, we must use (i), since only (i) has a formula for select mortality at the age of selection (without a $+1$ outside the brackets). $q_{[67]} = 0.75q_{[66]+1} = (0.75)(0.8)q_{67}$, with the second equality coming from (ii). So

$$q_{[67]} = 0.6\left(1 - \frac{8000}{8200}\right) = 0.014634$$

We can now calculate $l_{[67]}$.

$$l_{[67]} = \frac{l_{69}}{p_{[67]+1}p_{[67]}} = \frac{7700}{(1-0.03)(1-0.014634)} = \boxed{8056} \qquad \textbf{(C)}$$

8.31.

(a)

$$\begin{aligned}
p_{[85]} &= 1 - \frac{d_{[85]}}{l_{[85]}} = 1 - \frac{100}{1000} = 0.9 \\
p_{[86]} &= 1 - \frac{d_{[86]}}{l_{[86]}} = 1 - \frac{100}{850} = 0.882353 \\
l_{86} &= l_{[85]} - d_{[85]} = 1000 - 100 = 900 \\
l_{87} &= l_{[86]} - d_{[86]} = 850 - 100 = 750 \\
p_{86} &= \frac{l_{87}}{l_{86}} = \frac{750}{900} = 0.833333
\end{aligned}$$

(b)

$$\begin{aligned}
\int_2^\infty {}_tp_{[85]}\,\mathrm{d}t &= \int_0^\infty {}_tp_{[85]}\,\mathrm{d}t - \int_0^1 {}_tp_{[85]}\,\mathrm{d}t - \int_1^2 {}_tp_{[85]}\,\mathrm{d}t \\
&= 5.225 - \int_0^1 (1 - 0.1t)\mathrm{d}t - \int_1^2 (0.9 - 0.15(t-1))\mathrm{d}t
\end{aligned}$$

The previous line was developed by noting that $l_{[85]} = 1000$, $l_{86} = 900$, $l_{87} = 750$, so by linear interpolation, ${}_tp_{[85]} = 1 - 0.1t$ for $t \le 1$ and $0.9 - 0.15(t-1)$ for $1 \le t \le 2$. Each integral equals the midpoint of the integrand times the integral's range, since the integrands form trapezoids. So the integral from 0 to 1 is $0.5(1 + 0.9) = 0.95$ and the integral from 1 to 2 is $0.5(0.9 + 0.75) = 0.825$.

$$\int_2^\infty {}_tp_{[85]}\,\mathrm{d}t = 5.225 - 0.95 - 0.825 = \boxed{3.45}$$

(c)

$$\begin{aligned}
\mathring{e}_{[86]} &= \int_0^1 {}_tp_{[86]}\mathrm{d}t + \int_1^\infty {}_tp_{[86]}\mathrm{d}t \\
&= \int_0^1 \big(1-(1-0.882353)t\big)\mathrm{d}t + p_{[86]}\int_0^\infty {}_tp_{87}\,\mathrm{d}t \\
&= 0.941176 + 0.882353\frac{\int_2^\infty {}_tp_{[85]}\,\mathrm{d}t}{{}_2p_{[85]}} \\
&= 0.941176 + 0.882353\frac{3.45}{p_{[85]}p_{86}} \\
&= 0.941176 + 0.882353\frac{3.45}{0.75} = \boxed{5}
\end{aligned}$$

8.32.

$$ {}_tp_{[x]} = \exp\left(-\int_0^t \mu_{[x]+u}\,\mathrm{d}u\right)$$

$$\begin{aligned}
\int_0^t \mu_{[x]+u}\,\mathrm{d}u &= \int_0^t 0.9^{5-u}(A+Bc^{x+u})\mathrm{d}u \\
&= \int_0^t 0.9^{5-u}A\,\mathrm{d}u + \int_0^t 0.9^5Bc^x\left(\frac{c}{0.9}\right)^u\mathrm{d}u \\
&= \frac{0.9^{5-u}A}{-\ln 0.9}\bigg|_0^t + 0.9^5Bc^x\left(\frac{(c/0.9)^u}{\ln(c/9)}\right)\bigg|_0^t \\
&= \frac{A}{\ln 0.9}(0.9^5-0.9^{5-t}) + \frac{0.9^5Bc^x}{\ln c-\ln 0.9}\left(\left(\frac{c}{0.9}\right)^t-1\right) \\
&= \frac{A}{\ln 0.9}(0.9^5-0.9^{5-t}) + \frac{0.9^{5-t}Bc^x}{\ln c-\ln 0.9}(c^t-0.9^t)
\end{aligned}$$

Negating and exponentiating, we get the desired expression.

Quiz Solutions

8-1. All mortality rates used for this calculation are select.

$$\begin{aligned}
q_{[45]} &= 0.5(0.004) = 0.002 \\
q_{[45]+1} &= 0.5(0.00431) = 0.002155
\end{aligned}$$

The answer is

$$ {}_{1|}q_{[45]} = (1-0.002)(0.002155) = \boxed{\mathbf{0.002151}}$$

8-2.

$$\begin{aligned}
{}_{2|3}q_{[41]} &= {}_2p_{[41]} - {}_5p_{[41]} \\
{}_2p_{[41]} &= (1-0.003)(1-0.006) = 0.991018 \\
{}_5p_{[41]} &= 0.991018(1-0.009)(1-0.015)(1-0.018) = 0.949955
\end{aligned}$$

The answer is ${}_{2|3}q_{[41]} = 0.991018 - 0.949955 = \boxed{\mathbf{0.041063}}$.

8-3. We must go from $l_{[80]}$ to l_{83}, then back to $l_{[81]}$.

$$\begin{aligned}
q_{[80]} &= 0.5(0.1) = 0.05\\
q_{[80]+1} &= 0.8(0.2) = 0.16\\
l_{83} &= 1000(1-0.05)(1-0.16)(1-0.3) = 558.6\\
q_{[81]} &= 0.5(0.2) = 0.1\\
q_{[81]+1} &= 0.8(0.3) = 0.24\\
l_{[81]} &= \frac{558.6}{(1-0.1)(1-0.24)} = \boxed{\mathbf{816.67}}
\end{aligned}$$

Lesson 9

Supplementary Questions: Survival Distributions

9.1. You are given:

(i) For a cohort of 100 newly born children, the force of mortality is constant and equal to 0.01.
(ii) Birthday cards are sent each year to all lives in the cohort beginning on their 80^{th} birthdays, for as long as they live.

Determine the expected number of birthday cards each member of this cohort receives.

(A) 44.7 (B) 44.9 (C) 45.2 (D) 45.5 (E) 45.7

9.2. You are given:

(i) The probability that a milk carton on the shelf is purchased on any day is 20%.
(ii) Milk cartons are discarded after being on the shelf for 7 days.

Determine the average number of full days a purchased milk carton is on the shelf.

(A) 1.69 (B) 1.73 (C) 2.14 (D) 2.18 (E) 2.63

9.3. For a mortality table, you are given

(i) Uniform distribution of deaths is assumed between integral ages.
(ii) $\mu_{30.25} = 1$
(iii) $\mu_{30.5} = \frac{4}{3}$

Determine $\mu_{30.75}$.

(A) $\frac{5}{3}$ (B) 2 (C) $\frac{7}{3}$ (D) $\frac{5}{2}$ (E) 3

9.4. For (40), you are given

(i) $\mu_{40+t} = 1/\big(2(60-t)\big), \quad t < 60$
(ii) $\int_0^n {}_tp_{40}\,dt = 35$

Determine n.

(A) 42.0 (B) 45.0 (C) 47.5 (D) 50.0 (E) 52.5

9.5. You are given:

(i) $\mathring{e}_{30:\overline{20}|} = 19$
(ii) ${}_{20}p_{30} = 0.9$
(iii) Mortality between ages 50 and 100 follows $l_x = 1000(100 - x)$, $50 \le x \le 100$.

Calculate $\mathring{e}_{30:\overline{30}|}$.

(A) 26.2 (B) 27.1 (C) 27.6 (D) 28.0 (E) 28.5

9.6. T_0, the future lifetime of (0), has the following density function $f_0(t)$:
$g(t)$ follows the underlying mortality of the Illustrative Life Table.
$h(t) = 0.01$, $0 \le t \le 100$.

$$f_0(t) = \begin{cases} 1.2g(t) & 0 \le t \le 50 \\ kh(t) & t > 50 \end{cases}$$

Calculate ${}_{10}p_{45}$.

(A) 0.869 (B) 0.872 (C) 0.874 (D) 0.876 (E) 0.879

9.7. For (x), you are given

(i) $\mu_x = \alpha/(100 - x)$
(ii) ${}_{3}q_{33} = 0.0030$

Determine α.

(A) 0.064 (B) 0.066 (C) 0.068 (D) 0.070 (E) 0.072

9.8. You are given

$$\mu_x = \frac{1}{(100 - x)^2} \qquad 0 < x < 100$$

Calculate $f_0(95)$, the probability density function of survival at age 95.

(A) 0.031 (B) 0.033 (C) 0.036 (D) 0.038 (E) 0.040

9.9. You are given

(i) x is an integer.
(ii) ${}_{0.3}q_{x+0.2} = 0.1$
(iii) ${}_{s}q_{x+0.5} = 0.1$
(iv) Deaths are uniformly distributed between integral ages.

Determine s.

(A) 0.26 (B) 0.27 (C) 0.28 (D) 0.29 (E) 0.30

9.10. For a group of lives, $\mu_x = K$, where K is constant for each life. The distribution of K over all lives has probability density function

$$f_K(k) = 100e^{-100k} \qquad k > 0$$

Calculate ${}_{4|4}q_{44}$ for a life selected randomly from this group.

(A) 0.0356 (B) 0.0369 (C) 0.0376 (D) 0.0385 (E) 0.0392

Exercises continue on the next page . . .

9.11. You are given:

(i) $l_x = 100(120 - x)$ for $x \leq 90$
(ii) $\mu_x = \mu$ for $x > 90$
(iii) $\mathring{e}_{80}$ has the same value that it would have if $l_x = 100(120 - x)$ for $x \leq 120$.

Determine μ.

(A) 0.04 (B) 0.05 (C) 0.067 (D) 0.075 (E) 0.083

9.12. The force of mortality for an individual is

$$\mu_x = \frac{1}{3(120 - x)} \qquad 0 \leq x < 120$$

Calculate $\mathrm{Var}(T_{50})$.

(A) 314.27 (B) 393.75 (C) 628.54 (D) 787.50 (E) 1378.13

9.13. You are given

(i) $j(x) = 1.1^{x-100}$
(ii) $\mu_x = j(x)/\bigl(1 + j(x)\bigr)$

Calculate q_{103}/q_{102}.

(A) 0.95 (B) 0.98 (C) 1.02 (D) 1.03 (E) 1.04

9.14. In an actuarial student program, the number of years students stay in the program is distributed as follows:

1 year	0.85
2 years	0.60
3 years	0.55
4 years	0.45

The distribution of the amount of time in the program after 4 years has probability density function $f(t) = \mu e^{-\mu t}$, with μ selected to match the 0.45 probability of staying in the program 4 years.

Determine the average number of full years that students stay in the program.

(A) 2.8 (B) 3.6 (C) 4.5 (D) 4.6 (E) 4.7

Solutions

9.1. **[Lesson 5]** The probability of receiving a birthday card at birthday $k \geq 80$ is ${}_kp_0$, and

$$\sum_{k=80}^{\infty} {}_kp_0 = \sum_{k=0}^{\infty} e^{-0.8-0.01k} = \frac{e^{-0.8}}{1 - e^{-0.01}} = \boxed{45.1579} \qquad \textbf{(C)}$$

9.2. **[Lesson 5]** The average number of full days that a milk carton "survives" on the shelf is $e_{0:\overline{7}|}$. The t-day survival probability is ${}_tp_0 = 0.8^t$ for $t \le 7$.

$$e_{0:\overline{7}|} = \sum_{k=1}^{7} 0.8^k = \frac{0.8 - 0.8^8}{1 - 0.8} = 3.16114$$

Now we remove the contribution of cartons on the shelf 7 full days, $7(0.8^7)$, and divide by the probability of being purchased, $1 - 0.8^7$.

$$\frac{3.16114 - 7(0.8^7)}{1 - 0.8^7} = \boxed{\mathbf{2.1424}} \qquad \textbf{(C)}$$

9.3. **[Lesson 7]** Under UDD, $\mu_{x+s} = \frac{q_x}{1-sq_x}$. You can back out q_{30} using either $\mu_{30.25}$ or $\mu_{30.5}$; you don't need both.

$$\begin{aligned}
\mu_{30.25} = 1 &= \frac{q_{30}}{1 - 0.25q_{30}} \\
q_{30} &= 1 - 0.25q_{30} \\
1.25q_{30} &= 1 \\
q_{30} &= 0.8
\end{aligned}$$

or

$$\begin{aligned}
\mu_{30.5} = \frac{4}{3} &= \frac{q_{30}}{1 - 0.5q_{30}} \\
3q_{30} &= 4 - 2q_{30} \\
5q_{30} &= 4 \\
q_{30} &= 0.8
\end{aligned}$$

Therefore, the force of mortality at 30.75 is

$$\mu_{30.75} = \frac{0.8}{1 - 0.75(0.8)} = \boxed{\mathbf{2}} \qquad \textbf{(B)}$$

9.4. **[Lesson 5]** We calculate ${}_tp_{40}$. Since mortality is beta with $\alpha = 0.5$ and $\omega - 40 = 60$,

$${}_tp_{40} = \left(\frac{60 - t}{60}\right)^{0.5} \qquad t \le 60$$

Then

$$\begin{aligned}
35 &= \int_0^n \frac{(60 - t)^{0.5}\mathrm{d}t}{60^{0.5}} \\
&= \frac{60^{1.5} - (60 - n)^{1.5}}{1.5(60^{0.5})} \\
52.5 &= 60 - \frac{(60 - n)^{1.5}}{60^{0.5}} \\
60 - n &= \left(7.5(60^{0.5})\right)^{2/3} = 7.5^{2/3} \cdot 60^{1/3} = 7.5(8^{1/3}) = 15 \\
n &= \boxed{\mathbf{45}} \qquad \textbf{(B)}
\end{aligned}$$

9.5. **[Lesson 6]** We use the recursive formula (6.1).

$$\begin{aligned}\mathring{e}_{30:\overline{30}|} &= \mathring{e}_{30:\overline{20}|} + {}_{20}p_{30}\,\mathring{e}_{50:\overline{10}|}\\ &= 19 + 0.9\mathring{e}_{50:\overline{10}|}\end{aligned}$$

To calculate $\mathring{e}_{50:\overline{10}|}$ under uniform mortality: it is 5 times the probability of dying within 10 years plus 10 times the probability of surviving 10 years:

$$\mathring{e}_{50:\overline{10}|} = 5\left(\frac{10}{50}\right) + 10\left(\frac{40}{50}\right) = 9$$

Then

$$\mathring{e}_{30:\overline{30}|} = 19 + 0.9(9) = \boxed{27.1} \qquad \textbf{(B)}$$

9.6. **[Lesson 4]** From ages 45 to 50, $F_0(t) = 1.2G(t)$ where $G(t)$ is the cumulative distribution function of the Illustrative Life Table. The cumulative distribution function is the proportion of lives that died by an age, and can be computed as $G(t) = 1 - l_t/l_0$, so (using l's for values of the Illustrative Life Table)

$$\begin{aligned}F_0(45) &= 1.2\left(1 - \frac{l_{45}}{l_0}\right) = 1.2\left(1 - \frac{9{,}164{,}051}{10{,}000{,}000}\right) = 0.100314\\ F_0(50) &= 1.2\left(1 - \frac{l_{50}}{l_0}\right) = 1.2\left(1 - \frac{8{,}950{,}901}{10{,}000{,}000}\right) = 0.125892\\ {}_5p_{45} &= \frac{S_0(50)}{S_0(45)} = \frac{1 - 0.125892}{1 - 0.100314} = 0.97157\end{aligned}$$

At ages 50 and above the survival function is uniformly distributed with limiting age 100. Therefore ${}_tp_{50} = (50 - t)/50$, so

$${}_5p_{50} = \frac{45}{50} = 0.9$$

The 10-year probability of survival of (45) is ${}_{10}p_{45} = (0.97157)(0.9) = \boxed{0.87441}$. **(C)**

9.7. **[Lesson 3]** ${}_3p_{33} = 0.997$, and for this beta, ${}_tp_x = \big((100 - x - t)/(100 - x)\big)^{\alpha}$, so

$$\begin{aligned}\left(\frac{100 - 33 - 3}{100 - 33}\right)^{\alpha} &= 0.997\\ \left(\frac{64}{67}\right)^{\alpha} &= 0.997\\ \alpha &= \frac{\ln 0.997}{\ln(64/67)} = \boxed{0.065587} \qquad \textbf{(B)}\end{aligned}$$

9.8. **[Lesson 3]** The density function is $f_0(x) = {}_xp_0\,\mu_x$.

$$\begin{aligned}{}_xp_0 &= \exp\left(-\int_0^x \frac{\mathrm{d}u}{(100 - u)^2}\right)\\ &= \exp\left(-\frac{1}{100 - x} + \frac{1}{100}\right)\\ {}_{95}p_0 &= \exp\left(-\frac{1}{5} + \frac{1}{100}\right) = 0.826959\end{aligned}$$

$$f_0(95) = {}_{95}p_0\,\mu_{95} = (0.826959)\left(\frac{1}{5^2}\right) = \boxed{0.033078} \qquad \textbf{(B)}$$

9.9. **[Lesson 7]**

$$\begin{aligned}
{}_{0.3}q_{x+0.2} &= \frac{0.3q_x}{1-0.2q_x} = 0.1\\
0.3q_x &= 0.1 - 0.02q_x\\
q_x &= \frac{10}{32}\\
{}_sq_{x+0.5} &= \frac{sq_x}{1-0.5q_x} = 0.1\\
\frac{s(10/32)}{1-5/32} &= 0.1\\
\frac{10s}{27} &= 0.1\\
s &= \boxed{0.27} \qquad \textbf{(B)}
\end{aligned}$$

9.10. **[Lesson 3]** For each life, ${}_{4|4}q_{44} = {}_4p_{44} - {}_8p_{44}$ and ${}_tp_{44} = e^{-tK}$, so ${}_{4|4}q_{44} = e^{-4K} - e^{-8K}$. We integrate this over K's distribution.

$$\begin{aligned}
{}_{4|4}q_{44} &= \int_0^\infty \left(e^{-4K} - e^{-8K}\right)\left(100e^{-100K}\right)\mathrm{d}K\\
&= 100\int_0^\infty \left(e^{-104K} - e^{-108K}\right)\mathrm{d}K\\
&= 100\left(\frac{1}{104} - \frac{1}{108}\right) = \boxed{0.035613} \qquad \textbf{(A)}
\end{aligned}$$

9.11. **[Lessons 5 and 6]** By the recursive formula (6.1)

$$\mathring{e}_{80} = \mathring{e}_{80:\overline{10}|} + {}_{10}p_{80}\,\mathring{e}_{90}$$

Since mortality for the first ten years is unchanged, $\mathring{e}_{80:\overline{10}|}$ and ${}_{10}p_{80}$ are unchanged. We only need to equate $\mathring{e}_{90}$ between the two mortality assumptions. Under uniform mortality through age 120, $\mathring{e}_{90} = 15$, whereas under constant force mortality, $\mathring{e}_{90} = 1/\mu$. We conclude that $1/\mu = 15$, or $\mu = \frac{1}{15} = \boxed{0.066667}$. **(C)**

9.12. **[Lesson 5]** Lifetime follows a beta distribution with $\alpha = 1/3$ and $\omega = 120$, so future lifetime at 50 follows a beta distribution with $\alpha = 1/3$ and $\omega - x = 70$. Mean future lifetime is

$$\mathring{e}_{50} = \frac{\omega - x}{\alpha + 1} = \frac{70}{1 + 1/3} = 52.5$$

We will use formula (5.3) to evaluate the second moment.

$$\begin{aligned}
\mathbf{E}[T_{50}^2] &= 2\int_0^{70} t\,{}_tp_{50}\mathrm{d}t\\
&= 2\int_0^{70} t\left(\frac{(70-t)^{1/3}}{70^{1/3}}\right)\mathrm{d}t\\
&= \frac{2}{70^{1/3}}I
\end{aligned}$$

where $I = \int_0^{70} t(70-t)^{1/3}\mathrm{d}t$. We'll evaluate this integral using integration by parts.

$$\begin{aligned} I &= -\frac{3}{4}t(70-t)^{4/3}\bigg|_0^{70} + \frac{3}{4}\int_0^{70}(70-t)^{4/3}\mathrm{d}t \\ &= \left(\frac{3}{4}\right)\left(\frac{3}{7}\right)(70^{7/3}) \end{aligned}$$

So

$$\begin{aligned} \mathbf{E}[T_{50}^2] &= 2\left(\frac{3}{4}\right)\left(\frac{3}{7}\right)\left(\frac{70^{7/3}}{70^{1/3}}\right) = 3150 \\ \operatorname{Var}(T_{50}) &= 3150 - 52.5^2 = \boxed{393.75} \qquad \textbf{(B)} \end{aligned}$$

9.13. [Lesson 3]

$$\begin{aligned} p_{102} &= \exp\left(-\int_0^1 \mu_{102+t}\,\mathrm{d}t\right) \\ &= \exp\left(-\int_0^1 \frac{1.1^{2+t}}{1+1.1^{2+t}}\mathrm{d}t\right) \\ \int_0^1 \left(\frac{1.1^{2+t}}{1+1.1^{2+t}}\right)\mathrm{d}t &= \left(\frac{\ln(1+1.1^{2+t})}{\ln 1.1}\right)\bigg|_0^1 \\ &= \frac{\ln(1+1.1^3)-\ln(1+1.1^2)}{\ln 1.1} = 0.559278 \\ p_{102} &= e^{-0.559278} = 0.571622 \\ \int_0^1 \left(\frac{1.1^{3+t}}{1+1.1^{3+t}}\right)\mathrm{d}t &= \frac{\ln(1+1.1^4)-\ln(1+1.1^3)}{\ln 1.1} = 0.582616 \\ p_{103} &= e^{-0.582616} = 0.558435 \\ \frac{q_{103}}{q_{102}} &= \frac{1-0.558435}{1-0.571622} = \boxed{1.0308} \qquad \textbf{(D)} \end{aligned}$$

9.14. [Lesson 5] The life expectancy at entry, which we'll call age 0, is $e_0 = \sum_{k=1}^{\infty} {}_kp_0$. After four years, the survival rate is 0.45, and based on the given $f(t)$ is also

$$S_0(4) = \int_4^{\infty} f(t)\,\mathrm{d}t = \int_4^{\infty} \mu e^{-\mu t}\mathrm{d}t = e^{-4\mu}$$

Therefore $e^{-4\mu} = 0.45$, or $\mu = -0.25\ln 0.45$. And for $k \geq 4$:

$${}_kp_0 = \int_k^{\infty} \mu e^{-\mu t}\mathrm{d}t = e^{-k\mu} = 0.45^{k/4}$$

So e_0 is

$$\begin{aligned} e_0 &= 0.85 + 0.60 + 0.55 + \sum_{k=4}^{\infty} 0.45^{k/4} \\ &= 2 + \frac{0.45}{1-0.45^{1/4}} \end{aligned}$$

We've summed up the geometric series as $a/(1-r)$, where the first term a is 0.45 and the ratio of terms r is $0.45^{1/4}$.

$$e_0 = 2 + 2.4867 = \boxed{\mathbf{4.4867}} \qquad \textbf{(C)}$$

Practice Exams

Practice Exam 1

SECTION A — Multiple-Choice

1. A life age 60 is subject to Gompertz's law with $B = 0.001$ and $c = 1.05$.

Calculate $e_{60:\overline{2}|}$ for this life.

(A) 1.923 (B) 1.928 (C) 1.933 (D) 1.938 (E) 1.943

2. For a fully discrete 20-year deferred whole life insurance of 1000 on (50), you are given:

(i) Premiums are payable for 20 years.
(ii) The net premium is 12.
(iii) Deaths are uniformly distributed between integral ages.
(iv) $i = 0.1$
(v) ${}_{9}V = 240$ and ${}_{9.5}V = 266.70$.

Calculate ${}_{10}V$, the net premium reserve at the end of year 10.

(A) 272.75 (B) 280.00 (C) 281.40 (D) 282.28 (E) 282.86

3. For an annual premium 2-year term insurance on (60) with benefit b payable at the end of the year of death, you are given

(i)

t	p_{60+t-1}
1	0.98
2	0.96

(ii) The annual net premium is 25.41.
(iii) $i = 0.05$.

Determine the revised annual net premium if an interest rate of $i = 0.04$ is used.

(A) 25.59 (B) 25.65 (C) 25.70 (D) 25.75 (E) 25.81

4. In a double-decrement model, with decrements (1) and (2), you are given, for all $t > 0$:

(i) ${}_{t}p_x'^{(1)} = 10/(10+t)$
(ii) ${}_{t}p_x'^{(2)} = \big(10/(10+t)\big)^3$

Determine $q_x^{(1)}$.

(A) 0.068 (B) 0.074 (C) 0.079 (D) 0.083 (E) 0.091

5. A Type A universal life policy with death benefit 10,000 is sold to a person age 75. You are given the following information concerning charges and credits:

(i) 20% of premium is charged at the beginning of the first year.
(ii) The COI charge in the first year is based on $q_{75} = 0.02$.
(iii) Interest is credited on the account value at 4.5% effective.
(iv) A different interest rate is used to discount the COI.
(v) The account value is updated annually.

The policyholder contributes 1000 initially. At the end of the first year, the account value is 644.30.

Determine the interest rate used to discount the COI.

(A) 0.020 (B) 0.022 (C) 0.024 (D) 0.026 (E) 0.028

6. In a three-state Markov chain, you are given the following forces of transition:

$$\mu_t^{01} = 0.05 \qquad \mu_t^{10} = 0.04 \qquad \mu_t^{02} = 0.03 \qquad \mu_t^{12} = 0.10$$

All other forces of transition are 0.

Calculate the probability of an entity in state 0 at time 0 transitioning to state 1 before time 5 and staying there until time 5, then transitioning to state 0 before time 10 and staying there until time 10.

(A) 0.017 (B) 0.018 (C) 0.019 (D) 0.020 (E) 0.021

7. For a temporary life annuity-due of 1 per year on (30), you are given:

(i) The annuity makes 20 certain payments.
(ii) The annuity will not make more than 40 payments.
(iii) Mortality follows the Illustrative Life Table.
(iv) $i = 0.06$

Determine the expected present value of the annuity.

(A) 14.79 (B) 15.22 (C) 15.47 (D) 15.63 (E) 16.06

Exam questions continue on the next page . . .

8. For a fully discrete whole life insurance on (35) with face amount 100,000, you are given the following assumptions and experience for the fifth year:

	Assumptions	Actual
q_{39}	0.005	0.006
Surrender probability	0.05	0.06
Annual expenses	20	30
Settlement expenses—death	100	80
Settlement expenses—surrender	50	40
i	0.05	0.045

You are also given:

(i) The gross premium is 1725.
(ii) Reserves are gross premium reserves.
(iii) The gross premium reserve at the end of year 4 is 6000.
(iv) The cash surrender value for the fifth year is 6830.
(v) The surrender probability is based on the multiple-decrement table.

The fifth year gain is analyzed in the order of interest, surrender, death, expense.

Determine the fifth year surrender gain.

(A) −7.9 (B) −7.7 (C) −7.5 (D) 7.7 (E) 7.9

9. For a defined benefit pension plan, you are given

(i) Accrual rate is 1.6%
(ii) The pension benefit is a monthly annuity-due payable starting at age 65, based on final salary.
(iii) No benefits are payable for death in service.
(iv) There are no exits other than death before retirement.
(v) Salaries increase 3% per year.
(vi) $i = 0.04$

An employee enters the plan at age 32. At age 45, the accrued liability for the pension, using the projected unit credit method, is 324,645.

Calculate the normal contribution for this employee for the year beginning at age 45.

(A) 24,000 (B) 25,000 (C) 26,000 (D) 27,000 (E) 28,000

10. For an insurance with face amount 100,000, you are given:

(i)

$$\frac{\mathrm{d}}{\mathrm{d}t}{}_tV = 100$$

(ii) $P = 1380$
(iii) $\delta = 0.05$
(iv) $\mu_{x+t} = 0.03$

Determine ${}_tV$.

(A) 21,000 (B) 21,500 (C) 22,000 (D) 22,500 (E) 23,000

Exam questions continue on the next page . . .

11. A life age 90 is subject to mortality following Makeham's law with $A = 0.0005$, $B = 0.0008$, and $c = 1.07$.

Curtate life expectancy for this life is 6.647 years.

Using Woolhouse's formula with three terms, compute complete life expectancy for this life.

(A) 7.118 (B) 7.133 (C) 7.147 (D) 7.161 (E) 7.176

12. For a fully continuous whole life insurance of 1000 on (x):

(i) The gross premium is paid at an annual rate of 25.
(ii) The variance of future loss is 2,000,000.
(iii) $\delta = 0.06$

Employees are able to obtain this insurance for a 20% discount.

Determine the variance of future loss for insurance sold to employees.

(A) 1,281,533 (B) 1,295,044 (C) 1,771,626 (D) 1,777,778 (E) 1,825,013

13. You are given the following profit test for a 10-year term insurance of 100,000 on (x):

t	${}_{t-1}V$	P	E_t	I_t	bq_{x+t-1}	$p_{x+t-1}\,{}_tV$
0			−350			
1	0	1000	0	60.0	500	447.75
2	450	1000	20	85.8	600	795.20
3	800	1000	20	106.8	700	1092.30
4	1100	1000	20	124.8	800	1289.60
5	1300	1000	20	136.8	900	1412.18
6	1425	1000	20	144.3	1000	1435.50
7	1450	1000	20	145.8	1100	1285.70
8	1300	1000	20	136.8	1200	1037.40
9	1050	1000	20	121.8	1300	641.55
10	650	1000	20	97.8	1400	0.00

Which of the following statements is true?

I. The interest rate used in the calculation is $i = 0.06$.

II. At time 5, the reserve per survivor is 1425.

III. The profit signature component for year 3 is 92.81

(A) I and II only (B) I and III only (C) II and III only (D) I, II, and III
(E) The correct answer is not given by (A) , (B) , (C) , or (D) .

Exam questions continue on the next page . . .

14. Your company sells whole life insurance policies. At a meeting with the Enterprise Risk Management Committee, it was agreed that you would limit the face amount of the policies sold so that the probability that the present value of the benefit at issue is greater than 1,000,000 is never more than 0.05.

You are given:

(i) The insurance policies pay a benefit equal to the face amount b at the moment of death.
(ii) The force of mortality is $\mu_x = 0.001(1.05^x)$, $x > 0$
(iii) $\delta = 0.06$

Determine the largest face amount b for a policy sold to a purchaser who is age 45.

(A) 1,350,000 (B) 1,400,000 (C) 1,450,000 (D) 1,500,000 (E) 1,550,000

15. A Type A universal life policy with face amount 20,000 is issued to (50). The policy has a no-lapse guarantee, and remains in force as long as the policyholder pays a premium of 500 at the beginning of each year.

At time 10, the account value is 0, and the no-lapse guarantee is effective. The following assumptions are used for calculating the reserve:

(i) Mortality follows the Illustrative Life Table.
(ii) $i = 0.06$.
(iii) Expenses are 3% of premium plus 10, paid at the beginning of each year.
(iv) Death benefits are paid at the end of the year.

Calculate the gross premium reserve.

(A) 1992 (B) 2020 (C) 2042 (D) 2065 (E) 2089

16. For two lives (50) and (60) with independent future lifetimes:

(i) $\mu_{50+t} = 0.002t, \quad t > 0$
(ii) $\mu_{60+t} = 0.003t, \quad t > 0$

Calculate ${}_{20}q^{\,1}_{50:60} - {}_{20}q^{\,\,2}_{50:60}$.

(A) 0.17 (B) 0.18 (C) 0.30 (D) 0.31 (E) 0.37

17. You are given that $\mu_x = 0.002x + 0.005$.

Calculate ${}_{5|}q_{20}$.

(A) 0.015 (B) 0.026 (C) 0.034 (D) 0.042 (E) 0.050

18. For a 30-pay whole life insurance policy of 100,000 on (45), you are given:

(i) Benefits are payable at the end of the year of death.
(ii) Premiums and expenses are payable at the beginning of the year.
(iii) $\ddot{a}_{45} = 14.1121$
(iv) $\ddot{a}_{45:\overline{30}|} = 13.3722$
(v) $i = 0.06$
(vi) Expenses are:

	Per Premium	Per Policy
First Year	40%	200
Renewal Years	10%	r
Settlement		100

(vii) The gross premium determined by the equivalence principle is 1777.98.

Determine r.

(A) 37 (B) 38 (C) 39 (D) 40 (E) 41

19. For a special fully discrete whole life insurance on (40), you are given:

(i) The annual net premium in the first 20 years is $1000P_{40}$.
(ii) The annual net premium changes at age 60.
(iii) The death benefit is 1000 in the first 20 years, after which it is 2000.
(iv) Mortality follows the Illustrative Life Table.
(v) $i = 0.06$

Determine ${}_{21}V$, the net premium reserve for the policy at the end of 21 years.

(A) 282 (B) 286 (C) 292 (D) 296 (E) 300

20. You are given the following yield curve:

$$y_t = \begin{cases} 0.01 + 0.004t & 0 < t \le 5 \\ 0.02 + 0.002t & 5 \le t \le 20 \\ 0.06 & t \ge 20 \end{cases}$$

Calculate the 2-year forward rate on a 10-year zero-coupon bond.

(A) 0.040 (B) 0.044 (C) 0.047 (D) 0.049 (E) 0.052

Exam questions continue on the next page . . .

SECTION B — Written-Answer

1. (11 points) A special 5-year term insurance on (55) pays 1000 plus the net premium reserve at the end of the year of death. A single premium is paid at inception. You are given:

(i) Mortality follows the Illustrative Life Table.

(ii) $i = 0.06$

(a) (2 points) Calculate the net single premium for this policy.

(b) (3 points) Using the recursive formula for reserves, calculate net premium reserves for the policy at times 1, 2, 3, and 4.

(c) (2 points) Suppose the policy, in addition to paying death benefits, pays the single premium at the end of 5 years to those who survive.

Calculate the revised single premium.

(d) (2 points) Calculate the net single premium for an otherwise similar policy that pays 1000, but not the net premium reserve, at the end of the year of death.

(e) (2 points) Calculate the net single premium for an otherwise similar policy that pays 1000 plus the net single premium, but not the net premium reserve, at the end of the year of death.

2. (7 points) For a Type B universal life policy on (50) with face amount 100,000:

(i) The following charges and credits are made to the policy:

1. Expense charge is 500 per year.
2. COI is based on $q_{50+t} = 0.01 + 0.001t$.
3. Interest is credited at $i = 0.04$.
4. Surrender charge in year t is $1200 - 200t$ for $t = 1, 2, 3, 4, 5$.
5. The account value is updated annually.

(ii) The following assumptions are made:

1. Expenses are 400 per year.
2. Mortality is $q'^{(\text{death})}_{50+t} = 0.009 + 0.001t$.
3. Surrender rate is $q'^{(\text{surrender})}_{50+t} = 0.06$ for all t, with all surrenders occurring at the end of the year.
4. Interest is earned at $i = 0.04$.
5. There are no settlement expenses.

(iii) The account value at time 4 is 10,000.

(iv) The reserve equals the account value.

(v) The policyholder does not pay a premium in the fifth year.

(a) (2 points) Calculate the account value at the end of year 5.

(b) (3 points) Calculate the expected profit in year 5 per policy issued.

(c) (2 points) The corridor factor in the fifth year is 1.57.

Determine the largest amount that the policyholder can pay at the beginning of year 5 without forcing the face amount to increase due to the corridor factor.

Exam questions continue on the next page . . .

3. (9 points) A one-year term life insurance on (x) pays 2000 at the moment of decrement 1 and 1000 at the moment of decrement 2. You are given

(i) $q_x'^{(1)} = 0.1$

(ii) $q_x'^{(2)} = 0.3$

(iii) $\delta = 0.04$

(a) (3 points) The decrements are uniform in the multiple decrement table.
Calculate the EPV of the insurance.

(b) (3 points) The decrements are uniform in the associated single decrement tables.
Calculate the EPV of the insurance.

(c) (3 points) The forces of decrement are constant.
Calculate the EPV of the insurance.

4. (8 points) A continuous whole life annuity on (60) pays 100 per year.

You are given:

(i) Mortality follows $l_x = 1000(100 - x)$, $0 \le x \le 100$.

(ii) $\delta = 0.05$.

(a) (2 points) Calculate the probability that the present value of payments on the annuity is greater than its net single premium.

Use the following information for (b) and (c):

In addition to the annuity payments, a death benefit of 1000 is paid at the moment of death if death occurs within the first ten years.

(b) (4 points) Calculate the probability that the present value of payments on the annuity (including the death benefit) is greater than its net single premium.

(c) (2 points) Calculate the minimum value of the present value of payments.

Exam questions continue on the next page . . .

5. (10 points) A special whole life insurance on (35) pays a benefit at the moment of death. You are given:

(i) The benefit for death in year k is $9000 + 1000k$, but never more than 20,000.
(ii) Mortality follows the Illustrative Life Table.
(iii) $i = 0.06$.
(iv) $1000(IA)^{1}_{35:\overline{10}|} = 107.98$
(v) Premiums are payable monthly.

(a) (2 points) Calculate the net single premium for the policy assuming uniform distribution of deaths between integral ages.

(b) (2 points) Calculate the net single premium for a whole life annuity-due annuity on (35) of 1 per month using Woolhouse's formula and approximating $\mu_x = -0.5(\ln p_{x-1} + \ln p_x)$.

(c) (1 point) Calculate the net premium payable monthly, using the assumptions and methods of parts (a) and (b).

(d) (3 points) Calculate the net premium reserve at time 10, using the same method as was used to calculate the net premium.

Suppose that instead of the benefit pattern of (i), the death benefit of the insurance is $11{,}000 - 1000k$, but never less than 1000.

(e) (2 points) Calculate the net single premium for the insurance, assuming uniform distribution of deaths between integral ages.

Exam questions continue on the next page . . .

6. (11 points) The ZYX Company offers a defined benefit pension plan with the following provisions:

- At retirement at age 65, the plan pays a monthly whole life annuity-due providing annual income that accrues at the rate of 1.5% of final salary up to 100,000 and 2% of the excess of final salary over 100,000 for each year of service.
- There is no early retirement.
- There are no other benefits.

The following assumptions are made:

(i) No employees exit the plan before retirement except by death.
(ii) Retirement occurs at the beginning of each year.
(iii) Pre-retirement mortality follows the Illustrative Life Table.
(iv) Salaries increase 3% each year.
(v) $i = 0.06$.
(vi) $\ddot{a}_{65}^{(12)} = 11$.

The ZYX Company has the following 3 employees on January 1, 2015:

Name	Exact Age	Years of Service	Salary in Previous Year
Cramer	55	20	120,000
Liu	35	5	50,000
Smith	50	10	100,000

(a) (3 points) Show that the actuarial liability using TUC is 267,000 to the nearest 1000. You should answer to the nearest 10.

(b) (3 points) Calculate the normal contribution for the year using TUC.

(c) (1 point) Calculate the replacement ratio for Cramer if he retires at age 65 and the salary increases follow assumptions.

(d) (2 points) Fifteen years later, Smith retires. Smith's salary increases have followed assumptions. Smith would prefer an annual whole life annuity-due.

Calculate the annual payment that is equivalent to the pension plan's monthly benefit using Woolhouse's formula to two terms.

(e) (2 points) On January 2, 2015, a pension consultant suggests that $q_{39} = 0.00244$ is a better estimate of mortality than the rate in the Illustrative Life Table. No other mortality rate changes are suggested.

Recalculate the actuarial liability under TUC as of January 1, 2015 using this new assumption.

Solutions to the above questions begin on page 1573.

Appendix A. Solutions to the Practice Exams

Answer Key for Practice Exam 1

1	E	6	A	11	A	16	B
2	D	7	C	12	C	17	D
3	C	8	E	13	A	18	D
4	C	9	B	14	A	19	B
5	A	10	B	15	E	20	D

Practice Exam 1

SECTION A — Multiple-Choice

1. **[Section 5.2]** By formula (4.2),

$$p_{60} = \exp\left(-0.001(1.05^{60})\left(\frac{0.05}{\ln 1.05}\right)\right) = 0.981040$$

$$_2p_{60} = \exp\left(-0.001(1.05^{60})\left(\frac{1.05^2-1}{\ln 1.05}\right)\right) = 0.961518$$

Then $e_{60:\overline{2}|} = 0.981040 + 0.961518 = \boxed{1.9426}$. **(E)**

2. **[Section 41.2]** We need to back out q_{59}. We use reserve recursion. Since the insurance is deferred, $1000q_{59}$ is not subtracted from the left side.

$$({}_9V + P)(1.1^{0.5}) = {}_{9.5}V(1 - 0.5q_{59})$$

$$252(1.1^{0.5}) = 266.70 - 133.35q_{59}$$

$$q_{59} = \frac{2.40017}{133.35} = 0.018$$

Then the net premium reserve at time 10 is, by recursion from time 9,

$$\frac{252(1.1)}{1-0.018} = \boxed{282.28} \qquad \textbf{(D)}$$

3. **[Lesson 24]** The revised premium for the entire policy is 25.41 times the ratio of the revised premium per unit at 4% to the original premium per unit at 5%.

We calculate the original net premium per unit, $P^{1}_{60:\overline{2}|}$.

$$\ddot{a}_{60:\overline{2}|} = 1 + \frac{0.98}{1.05} = 1.93333$$

$$A^{1}_{60:\overline{2}|} = \frac{0.02}{1.05} + \frac{(0.98)(0.04)}{1.05^2} = 0.054603$$

$$P^{1}_{60:\overline{2}|} = \frac{A^{1}_{60:\overline{2}|}}{\ddot{a}_{60:\overline{2}|}} = \frac{0.054603}{1.93333} = 0.028243$$

Now we recalculate at 4%. Call the revised premium $P'^{\,1}_{60:\overline{2}|}$.

$$\ddot{a}_{60:\overline{2}|} = 1 + \frac{0.98}{1.04} = 1.94231$$
$$A^{\,1}_{60:\overline{2}|} = \frac{0.02}{1.04} + \frac{(0.98)(0.04)}{1.04^2} = 0.055473$$
$$P'^{\,1}_{60:\overline{2}|} = \frac{0.055473}{1.94231} = 0.028561$$

So the revised premium for benefit b is $25.41(0.028561/0.028243) = \boxed{25.696}$. **(C)**

4. [Lesson 48]

$$
\begin{aligned}
{}_tp_x^{(\tau)} &= \left(\frac{10}{10+t}\right)\left(\frac{10}{10+t}\right)^3 = \left(\frac{10}{10+t}\right)^4 \\
\mu_{x+t}^{(1)} &= -\frac{\mathrm{d}\ln {}_tp_x'^{(1)}}{\mathrm{d}t} \\
&= -\frac{\mathrm{d}\big(\ln 10 - \ln(10+t)\big)}{\mathrm{d}t} \\
&= \frac{1}{10+t} \\
q_x^{(1)} &= \int_0^1 {}_tp_x^{(\tau)}\,\mu_{x+t}^{(1)}\,\mathrm{d}t \\
&= \int_0^1 \left(\frac{10}{10+t}\right)^4\left(\frac{1}{10+t}\right)\mathrm{d}t \\
&= \int_0^1 \frac{10^4\,\mathrm{d}t}{(10+t)^5} \\
&= -\left(\frac{10^4}{4}\right)\left(\frac{1}{(10+t)^4}\right)\Bigg|_0^1 \\
&= \left(\frac{10^4}{4}\right)\left(\frac{1}{10^4} - \frac{1}{11^4}\right) \\
&= \boxed{0.079247} \qquad \textbf{(C)}
\end{aligned}
$$

5. [Section 67.1] Use the formula relating account values. Let $v_q = 1/(1+i_q)$ be the discount factor for COI.

$$
\begin{aligned}
\mathrm{AV}_1 &= \frac{(P - E - v_q q_{75} FA)(1+i)}{1 - v_q(1+i)q_{75}} \\
644.30 &= \frac{(1000 - 200 - 200v_q)(1.045)}{1 - 1.045v_q(0.02)} \\
644.30 - 13.4659v_q &= 836 - 209v_q \\
195.5341v_q &= 191.7 \\
v_q &= 0.9803915 \\
i_q &= \frac{1}{0.9803915} - 1 = \boxed{0.02} \qquad \textbf{(A)}
\end{aligned}
$$

6. [Section 44.1] Let ${}_5p_0^{\overline{01}}$ be the probability that an entity in state 0 at time 0 transitions to state 1 before time 5 and stays there until time 5, and let ${}_5p_5^{\overline{10}}$ be the probability that an entity in state 1 at time 5 transitions to state 0 before time 10 and stays there until time 10. We'll use formula (44.9) for both transitions. Notice that the formula is the same with 0 and 1 switched, except that ${}_5p_0^{\overline{01}}$ uses $\mu^{01} = 0.05$ and ${}_5p_5^{\overline{10}}$ uses $\mu^{10} = 0.04$ outside the parentheses.

$$\frac{e^{-\mu^{0\bullet}t}}{\mu^{1\bullet} - \mu^{0\bullet}} + \frac{e^{-\mu^{1\bullet}t}}{\mu^{0\bullet} - \mu^{1\bullet}} = \frac{e^{-0.08(5)}}{0.14 - 0.08} + \frac{e^{-0.14(5)}}{0.08 - 0.14} = 2.89558$$

$${}_5p_0^{\overline{01}} = 0.05(2.89558) = 0.14478$$

$${}_5p_5^{\overline{10}} = 0.04(2.89558) = 0.11582$$

The answer is (0.14478)(0.11582) = **0.01677**. **(A)**

7. [Lesson 17] This annuity is the sum of a 20-year certain annuity-due and a 20-year deferred 20-year temporary life annuity due.

$$\ddot{a}_{\overline{20|}} = \frac{1 - (1/1.06)^{20}}{1 - 1/1.06} = 12.15812$$

$$\begin{aligned}
{}_{20|}\ddot{a}_{30:\overline{20|}} &= {}_{20|}\ddot{a}_{30} - {}_{40|}\ddot{a}_{30} \\
&= {}_{20}E_{30}\,\ddot{a}_{50} - {}_{40}E_{30}\,\ddot{a}_{70} \\
&= {}_{20}E_{30}\,\ddot{a}_{50} - {}_{20}E_{30}\,{}_{20}E_{50}\,\ddot{a}_{70} \\
&= (0.29374)(13.2668) - (0.29374)(0.23047)(8.5693) \\
&= 3.89699 - (0.067699)(8.5693) \\
&= 3.89699 - 0.58013 = 3.31686
\end{aligned}$$

The expected present value of the annuity is 12.15812 + 3.31686 = **15.4750**. **(C)**

8. [Lesson 68] Surrender gain per surrender is the ending reserve (which is released into profit) minus the benefit paid and minus expenses. The ending gross premium reserve is

$${}_5V = \frac{(6000 + 1725 - 20)(1.05) - (100{,}000 + 100)(0.005) - (6830 + 50)(0.05)}{1 - 0.05 - 0.005} = 7667.46$$

Using assumed expenses, the surrender gain per surrender is 7667.46 − (6830 + 50) = 787.46. The gain is (0.06 − 0.05)(787.46) = **7.8746**. **(E)**

9. [Section 61.4] Using PUC, if there are no exit benefits and accruals are the same percentage each year, the normal contribution is the initial accrued liability divided by the number of years of service, or 324,645/13 = **24,973**. **(B)**

10. [Section 41.3]

$$100 = (0.05 + 0.03)\,{}_tV + 1380 - 100{,}000(0.03) = 0.08\,{}_tV - 1620$$

$${}_tV = \frac{1720}{0.08} = \boxed{21{,}500} \qquad \textbf{(B)}$$

11. **[Section 22.2]** By equation (22.10),

$$\mathring{e}_x = e_x + \tfrac{1}{2} - \tfrac{1}{12}\mu_x$$

Force of mortality for (90) is $\mu_{90} = 0.0005 + 0.0008(1.07^{90}) = 0.353382$. Thus

$$\mathring{e}_{90} = 6.647 + 0.5 - \tfrac{1}{12}(0.353382) = \boxed{7.118} \qquad \textbf{(A)}$$

12. **[Lesson 30]** The variance of future loss for a gross premium of 25 is

$$\begin{aligned} 2{,}000{,}000 &= \operatorname{Var}\left(v^{T_x}\right)\left(1000 + \frac{25}{0.06}\right)^2 \\ &= \operatorname{Var}\left(v^{T_x}\right)(2{,}006{,}944) \end{aligned}$$

If we replace 25 with 20 (for a 20% discount) in the above formula, it becomes

$$\begin{aligned} \operatorname{Var}\left({}_0L\right) &= \operatorname{Var}\left(v^{T_x}\right)\left(1000 + \frac{20}{0.06}\right)^2 \\ &= \operatorname{Var}\left(v^{T_x}\right)(1{,}777{,}778) \end{aligned}$$

We see that this is 1,777,778/2,006,944 times the given variance, so the final answer is

$$\operatorname{Var}({}_0L) = \frac{1{,}777{,}778}{2{,}006{,}944}(2{,}000{,}000) = \boxed{1{,}771{,}626} \qquad \textbf{(C)}$$

13. **[Lesson 65]**

I From the row for year 1, with 0 reserves and expenses, we see that $I_t/P_t = 0.06$, so the interest rate is 0.06. ✓

II Looking at the line for $t = 6$, we see that the reserve per survivor to time $t - 1 = 5$ is 1425. ✓

III First, the profit in year 3 is $800 + 1000 - 20 + 106.8 - 700 - 1092.3 = 94.50$. We deduce survivorship from the bq_{x+t-1} column, and we see that the mortality rates in the first two years are 0.005 and 0.006, so the profit signature component of year'3 is $(0.995)(0.994)(94.50) = 93.46$. ✗.

(A)

14. **[Lesson 13]** The present value of the benefit decreases with increasing survival time, so the 95th percentile of the present value of the insurance corresponds to the 5th percentile of survival time. The survival probability is

$$\begin{aligned} {}_tp_{45} &= \exp\left(-\int_0^t 0.001(1.05^{45+u})\,\mathrm{d}u\right) \\ -\ln {}_tp_{45} &= \left.\frac{0.001(1.05^{45+u})}{\ln 1.05}\right|_0^t \\ &= \frac{0.001(1.05^{45+t} - 1.05^{45})}{\ln 1.05} \end{aligned}$$

Setting ${}_tp_{45} = 0.95$,

$$\begin{aligned}\frac{0.001(1.05^{45+t} - 1.05^{45})}{\ln 1.05} &= -\ln 0.95\\ 1.05^{45+t} &= (-1000\ln 0.95)(\ln 1.05) + 1.05^{45} = 11.48762\\ 1.05^t &= \frac{11.48762}{1.05^{45}} = 1.27853\\ t &= \frac{\ln 1.27853}{\ln 1.05} = 5.0361\end{aligned}$$

The value of Z if death occurs at $t = 5.0361$ is $be^{-5.0361(0.06)}$, so the largest face amount is $1{,}000{,}000e^{5.0361(0.06)} =$ **1,352,786**. **(A)**

15. **[Section 67.1]** The expected present value of future benefits and expenses is

$$20{,}000A_{60} + \big(10 + 0.03(500)\big)\ddot{a}_{60} = 20(369.13) + 25(11.1454) = 7661.24$$

The expected present value of future premiums is $500\ddot{a}_{60} = 500(11.1454) = 5572.70$. The gross premium reserve is $7661.24 - 5572.70 =$ **2088.54**. **(E)**

16. **[Lesson 56]** ${}_{20}q^{\,1}_{50:60} - {}_{20}q^{\,2}_{50:60} = {}_{20}q_{50}\,{}_{20}p_{60}$, and

$$\begin{aligned}{}_{20}q_{50} &= 1 - \exp\left(-\int_0^{20} 0.002t\,\mathrm{d}t\right)\\ &= 1 - e^{-0.001(20)^2} = 1 - 0.670320 = 0.329680\\ {}_{20}p_{60} &= \exp\left(-\int_0^{20} 0.003t\,\mathrm{d}t\right)\\ &= e^{-0.0015(20)^2} = 0.548812\\ {}_{20}q_{50}\,{}_{20}p_{60} &= (0.329680)(0.548812) = \boxed{0.180932} \qquad \textbf{(B)}\end{aligned}$$

17. **[Lesson 3]** ${}_{5|}q_{20} = \big(S_0(25) - S_0(26)\big)/S_0(20)$, so we will calculate these three values of $S_0(x)$. (Equivalently, one could calculate ${}_5p_{20}$ and ${}_6p_{20}$ and take the difference.) The integral of μ_x is

$$\int_0^x \mu_u\,\mathrm{d}u = \left(\frac{0.002u^2}{2} + 0.005u\right)\bigg|_0^x = 0.001x^2 + 0.005x$$

so

$$\begin{aligned}S_0(20) &= \exp\Big(-\big(0.001(20^2) + 0.005(20)\big)\Big) = \exp(-0.5) = 0.606531\\ S_0(25) &= \exp\Big(-\big(0.001(25^2) + 0.005(25)\big)\Big) = \exp(-0.75) = 0.472367\\ S_0(26) &= \exp\Big(-\big(0.001(26^2) + 0.005(26)\big)\Big) = \exp(-0.806) = 0.446641\end{aligned}$$

and the answer is

$${}_{5|}q_{20} = \frac{0.472367 - 0.446641}{0.606531} = \boxed{0.042415} \qquad \textbf{(D)}$$

18. **[Lesson 28]** By the equivalence principle,

$$G(0.9\ddot{a}_{45:\overline{30}|} - 0.3) = 100{,}100A_{45} + ra_{45} + 200 \qquad (*)$$

$$1000A_{45} = 1000\,(1 - d\ddot{a}_{45}) = 1000\left(1 - \frac{0.06}{1.06}(14.1121)\right) = 201.2$$

$$a_{45} = 14.1121 - 1 = 13.1121$$

$$0.9\ddot{a}_{45:\overline{30}|} - 0.3 = 0.9(13.3722) - 0.3 = 11.7350$$

Substituting into (*),

$$1777.98(11.7350) = 100.1(201.2) + 13.1121r + 200$$

$$r = \frac{1777.98(11.7350) - 100.1(201.2) - 200}{13.1121} = \boxed{40} \qquad \textbf{(D)}$$

19. **[Lessons 36 and 39]** Because premiums and benefits are the same as for an insurance on (40) through year 20, ${}_{20}V$ must be the same as for a standard 1000 whole life insurance on (40), or

$${}_{20}V_{40} = 1 - \frac{\ddot{a}_{60}}{\ddot{a}_{40}} = 1 - \frac{11.1454}{14.8166} = 0.247776$$

Then by the equivalence principle, this reserve plus expected future net premiums must equal expected future benefits. If we let P be the premium after age 60:

$$2000A_{60} = 247.776 + P\ddot{a}_{60}$$

$$2000(0.36913) = 247.776 + P(11.1454)$$

$$P = \frac{2000(0.36913) - 247.776}{11.1454} = 44.0077$$

Now we roll the reserve forward one year.

$${}_{21}V = \frac{({}_{20}V + P)(1+i) - 2000q_{60}}{1 - q_{60}}$$

$$= \frac{(247.776 + 44.0077)(1.06) - 2000(0.01376)}{1 - 0.01376}$$

$$= \boxed{285.70} \qquad \textbf{(B)}$$

20. **[Lesson 62]**

$$y_2 = 0.018$$

$$y_{12} = 0.044$$

$$(1 + f(2,12))^{10} = \frac{1.044^{12}}{1.018^2} = 1.617746$$

$$f(2,12) = \sqrt[10]{1.617446} - 1 = \boxed{0.0493} \qquad \textbf{(D)}$$

SECTION B — Written-Answer

1. [Section 39.2]

(a) The reserve at time 5 is 0, so the single premium P is determined from

$$0 = P(1+i)^5 - 1000\sum_{k=1}^{5} q_{55+k-1}(1+i)^{5-k}$$

or

$$\begin{aligned}P &= 1000\sum_{k=1}^{5} q_{55+k-1}v^k\\ &= 1000\left(\frac{0.00896}{1.06}+\frac{0.00975}{1.06^2}+\frac{0.01062}{1.06^3}+\frac{0.01158}{1.06^4}+\frac{0.01262}{1.06^5}\right)\\ &= \boxed{\mathbf{44.6499}}\end{aligned}$$

(b) Because the net premium reserve is paid on death, the recursion does not divide by p_x.

$$\begin{aligned}44.6499(1.06) - 8.96 &= 38.3689\\ 38.3689(1.06) - 9.75 &= 30.9210\\ 30.9210(1.06) - 10.62 &= 22.1563\\ 22.1563(1.06) - 11.58 &= 11.9057\end{aligned}$$

Although not required, you could check the calculation by doing one more recursion: $11.9057(1.06) - 12.62 = 0$.

(c) The reserve at time 5 is P, so the single premium P is determined from

$$P = P(1+i)^5 - 1000\sum_{k=1}^{5} q_{55+k-1}(1+i)^{5-k}$$

or

$$P(1-v^5) = 1000\sum_{k=1}^{5} q_{55+k-1}v^k$$

We divide the answer to part (a) by $1-v^5$:

$$44.6499/(1-1/1.06^5) = \boxed{\mathbf{176.6621}}$$

(d)

$$1000A^{\,1}_{55:\overline{5|}} = 1000(A_{55} - {}_5E_{55}\,A_{60}) = 305.14 - (0.70810)(369.13) = \boxed{\mathbf{43.7590}}$$

(e)

$$\begin{aligned}P &= (1000+P)A^{\,1}_{55:\overline{5|}}\\ P &= \frac{43.7590}{1-0.0437590} = \boxed{\mathbf{45.7615}}\end{aligned}$$

2. [Section 67.2]

(a)

$$(10{,}000 - 500)(1.04) - 0.014(100{,}000) = \boxed{\mathbf{8480}}$$

(b) Calculate profit per policy in force at the beginning of year 5. Expected death benefit is 0.013(100,000+ 8,480) = 1410.24. Expected surrender benefit is (0.987)(0.06)(8,480 − 200) = 490.34. Expected ending account value is (0.987)(0.94)(8,480) = 7,867.57. So profit per policy in force at the beginning of year 5 is

$$(10{,}000 - 400)(1.04) - 1410.24 - 490.34 - 7867.57 = 215.84$$

Persistency to the beginning of year 4 is persistency from surrenders times persistency from deaths, or $(0.94^4)(0.991)(0.99)(0.989)(0.988) = 0.748468$. So profit per policy issued is $215.84(0.748468) = \boxed{\mathbf{161.55}}$.

(c) The death benefit must not be more than 1.57 times the account value:

$$\begin{aligned} 1.57\,\mathrm{AV}_5 &\le 100{,}000 + \mathrm{AV}_5 \\ \mathrm{AV}_5 &\le \frac{100{,}000}{0.57} = 175{,}439 \end{aligned}$$

AV_5 in terms of the premium P is

$$(10{,}000 + P - 500)(1.04) - 0.014(100{,}000) = \mathrm{AV}_5 = 175{,}439$$

It follows that the maximum P is

$$P = \frac{175{,}439 + 1{,}400}{1.04} - 9500 = \boxed{\mathbf{160{,}537}}$$

3. [Lessons 49 and 51]

(a)

$$\begin{aligned} p_x^{(\tau)} &= (0.9)(0.7) = 0.63 \\ q_x^{(1)} &= (0.37)\left(\frac{\ln 0.9}{\ln 0.63}\right) = 0.084373 \\ q_x^{(2)} &= (0.37)\left(\frac{\ln 0.7}{\ln 0.63}\right) = 0.285627 \end{aligned}$$

Since the decrements are uniform in the multiple decrement table, ${}_sp_x^{(\tau)}\,\mu_{x+s}^{(j)}$ is constant and equal to $q_x^{(j)}$. The EPV of the insurance is

$$\int_0^1 v^s\,{}_sp_x^{(\tau)}(2000\mu_{x+s}^{(1)} + 1000\mu_{x+s}^{(2)})\,\mathrm{d}s = \big(2000(0.084373) + 1000(0.285627)\big)\left(\frac{1 - e^{-0.04}}{0.04}\right) = \boxed{\mathbf{445.41}}$$

(b) The forces of mortality are $\mu_{x+s}^{(1)} = \frac{0.1}{1-0.1s}$ and $\mu_{x+s}^{(2)} = \frac{0.3}{1-0.3s}$. Also, ${}_sp_x^{(\tau)} = (1 - 0.1s)(1 - 0.3s)$. So the EPV of the insurance is

$$\mathrm{EPV} = \int_0^1 v^s(1 - 0.1s)(1 - 0.3s)\left(2000\frac{0.1}{1 - 0.1s} + 1000\frac{0.3}{1 - 0.3s}\right)\mathrm{d}s$$

$$= \int_0^1 e^{-0.04s}(500-90s)\,\mathrm{d}s$$

$$= -\frac{e^{-0.04s}}{0.04}(500-90s)\Big|_0^1 - 90\frac{\int_0^1 e^{-0.04s}\,\mathrm{d}s}{0.04}$$

$$= \frac{500-410e^{-0.04}}{0.04} - \frac{90(1-e^{-0.04})}{0.04^2} = \boxed{\mathbf{446.31}}$$

(c) The forces of decrement are $-\ln p_x'^{(j)}$, or $\mu_x^{(1)} = -\ln 0.9$ and $\mu_x^{(2)} = -\ln 0.7$. The probability of survival from both decrements under constant force is

$$ {}_sp_x^{(\tau)} = {}_sp_x'^{(1)}, {}_sp_x'^{(2)} = (0.9^s)(0.7^s) = 0.63^s$$

The EPV of the insurance is

$$\mathrm{EPV} = \int_0^1 v^s\, {}_sp_x^{(\tau)}\,(2000\mu_x^{(1)} + 1000\mu_x^{(2)})\mathrm{d}s$$

$$= \int_0^1 e^{(-0.04+\ln 0.63)s}\underbrace{(-2000\ln 0.9 - 1000\ln 0.7)}_{567.396}\mathrm{d}s$$

$$= 567.396\int_0^1 e^{(-0.04+\ln 0.63)s}\mathrm{d}s$$

$$= \frac{567.396}{-\ln 0.63 + 0.04}(1-0.63e^{-0.04}) = \boxed{\mathbf{446.09}}$$

4. [Lesson '20]

(a) First let's calculate the net single premium. We can ignore the 100 per year factor; it just scales up the numbers.

$$\bar{A}_{60} = \frac{1-e^{-0.05(40)}}{0.05(40)} = 0.432332$$

$$\bar{a}_{60} = \frac{1-0.432332}{0.05} = 11.35335$$

$\bar{a}_{\overline{T}|} = \bar{a}_{60}$ when:

$$\frac{1-e^{-0.05t}}{0.05} = \frac{1-0.432332}{0.05}$$

$$e^{-0.05t} = 0.432332$$

$$t = -\frac{\ln 0.432332}{0.05} = 16.77121$$

The probability that $T_{60} > 16.77121$ is $1 - 16.77121/40 = \boxed{\mathbf{0.58072}}$.

(b) First let's calculate the net single premium.

$$\bar{A}^{\,1}_{60:\overline{10}|} = \frac{1-e^{-0.05(10)}}{0.05(40)} = 0.196735$$

$$1000\bar{A}^{\,1}_{60:\overline{10}|} + 100\ddot{a}_{60} = 196.735 + 1135.335 = 1332.070$$

The present value of payments may be higher than 1332.070 in the first 10 years. However, let's begin by calculating the time $t > 10$ at which the present value of payments is higher than 1332.070.

$$\begin{aligned} 100\left(\frac{1-e^{-0.05t}}{0.05}\right) &= 1332.070 \\ e^{-0.05t} &= 0.333965 \\ t &= -\frac{\ln 0.333965}{0.05} = 21.93438 \end{aligned}$$

Now let's determine the time $t < 10$ for which the present value of payments is 1332.070.

$$\begin{aligned} 1000e^{-0.05t} + 100\left(\frac{1-e^{-0.05t}}{0.05}\right) &= 1332.07 \\ -1000e^{-0.05t} + 2000 &= 1332.07 \\ e^{-0.05t} &= 0.667930 \\ t &= -\frac{\ln 0.667930}{0.05} = 8.071437 \end{aligned}$$

Note that the present value of payments increases during the first 10 years. You see this from the second line above; $e^{-0.05t}$ has a negative coefficient and is a decreasing function of t, so the left side of the equation increases as t increases. Thus the present value of payments is greater than 1332.07 in the ranges $(8.071437, 10]$ and $(21.93438, \infty)$. The probability that death occurs in one of those ranges is $\bigl((10-8.071437)+(40-21.93438)\bigr)/40 =$ **0.49986**.

(c) For death right after time 10, the present value of the payments is

$$100\bar{a}_{\overline{10|}} = 100\left(\frac{1-e^{-0.5}}{0.05}\right) = 786.94$$

For death at time $t \le 10$, the present value of the payments is $2000 - 1000e^{-0.05t}$, which is always greater than 786.94. Therefore, **786.94** is the minimum loss.

5. [Section 22.2 and Lesson 26]

(a) The insurance can be expressed as a level whole life insurance of 9000, plus a 10-year increasing term insurance of 1000, plus a 10-year deferred insurance of 11,000. See figure A.1. Let A be the net single premium for the insurance payable at the end of the year of death.

$$\begin{aligned} A &= 9000A_{35} + 1000(IA)^{1}_{35:\overline{10|}} + 11{,}000\,{}_{10}E_{35}\,A_{45} \\ &= 9(128.72) + 107.98 + 11(0.54318)(201.20) = 2468.63 \end{aligned}$$

Multiplying by i/δ, we get $1.02971(2468.63) =$ **2541.97**.

(b)

$$\begin{aligned} \mu_{35} &\approx -0.5\ln(l_{36}/l_{34}) = -0.5\ln(9{,}401{,}688/9{,}438{,}571) = 0.0019577 \\ 12\ddot{a}^{(12)}_{35} &= 12\left(15.3926 - \tfrac{11}{24} - \tfrac{143}{1728}(0.0019577 + \ln 1.06)\right) = \mathbf{179.15} \end{aligned}$$

(c) $2541.97/179.15 =$ **14.1889**.

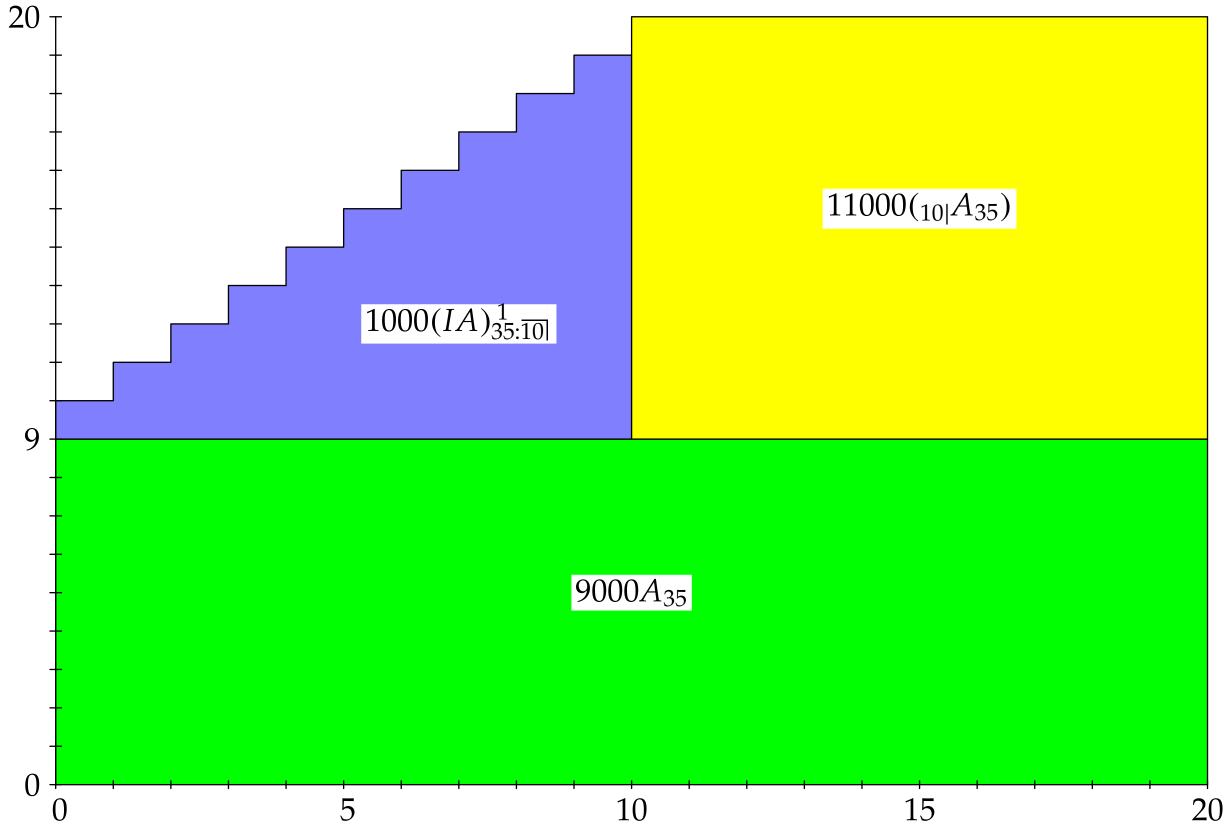


Figure A.1: Decomposition of increasing insurance in question 5

(d) We need to calculate $20{,}000\bar{A}_{45}$ and $\ddot{a}^{(12)}_{45}$.

$$\begin{aligned}
20{,}000\bar{A}_{45} &= 1.02971(20)(201.20) = 4143.55 \\
\mu_{45} &\approx -0.5\ln(l_{46}/l_{44}) = -0.5\ln(9{,}127{,}426/9{,}198{,}149) = 0.0038592 \\
12\ddot{a}^{(12)}_{45} &= 12\left(14.1121 - \tfrac{11}{24} - \tfrac{143}{1728}(0.0038592 + \ln 1.06)\right) = 163.78 \\
{}_{10}V &= 4143.55 - 14.1889(163.78) = \boxed{\mathbf{1819.64}}
\end{aligned}$$

(e) This insurance can be decomposed into a 10-year decreasing insurance plus a 10-year deferred whole life insurance. The EPV of the decreasing insurance can be derived from

$$(IA)^{1}_{35:\overline{10}|} + (DA)^{1}_{35:\overline{10}|} = 11A^{1}_{35:\overline{10}|}$$

Let A be the net single premium for the insurance payable at the end of the year of death.

$$\begin{aligned}
A &= 1000\left(11A^{1}_{35:\overline{10}|} - (IA)^{1}_{35:\overline{10}|}\right) + 1000\,{}_{10}E_{35}\,A_{45} \\
&= 1000\Big(11\big(0.12872 - (0.54318)(0.20120)\big) - 0.10798\Big) + 1000(0.54318)(0.20120) \\
&= 215.06
\end{aligned}$$

Multiplying by i/δ, we get $1.02971(215.06) = \boxed{\mathbf{221.45}}$.

An equivalent alternative is to evaluate the insurance as a whole life insurance for 11,000 minus a 10-year term increasing insurance for 1000 minus a 10-year deferred whole life insurance for 10,000.

6. [Lesson 61]

(a) For Cramer, ${}_{10}E_{55} = 0.48686$.

$$20\big(0.015(100{,}000) + 0.02(20{,}000)\big)(0.48686)(11) = 203{,}507.5$$

For Liu, ${}_{30}E_{35} = (0.54318)(0.25634) = 0.13924$.

$$5\big((0.015)(50{,}000)\big)(0.13924)(11) = 5743.6$$

For Smith, ${}_{15}E_{50} = (0.72137)(0.48686) = 0.35121$.

$$10(0.015)(100{,}000)(0.35121)(11) = 57{,}949.7$$

Total actuarial liability is $203{,}507.5 + 5743.6 + 57{,}949.7 = \boxed{267{,}201}$.

(b) For Cramer, salary will be $120,000(1.03) = 123{,}600$ next year. The discounted value of next year's liability is

$$21\big(0.015(100{,}000) + 0.02(23{,}600)\big)(0.48686)(11) = 221{,}780.3$$

For Liu, salary will not exceed 100,000, so we can calculate the normal contribution directly using formula (61.4):

$$5743.6\left(1.03\left(\frac{6}{5}\right) - 1\right) = 1355.5$$

For Smith, salary will be $100{,}000(1.03) = 103{,}000$ next year. The discounted value of next year's liability is

$$11\big(0.015(100{,}000) + 0.02(3{,}000)\big)(0.35121)(11) = 66{,}294.4$$

The normal contribution is $(221{,}780.3 - 203{,}507.5) + 1355.5 + (66{,}294.4 - 57{,}949.9) = \boxed{27{,}973}$.

(c) Final salary is $120{,}000(1.03^{10}) = 161{,}270$. Annual pension is

$$30\big(0.015(100{,}000) + 0.02(61{,}270)\big) = 81{,}762$$

The replacement ratio is $81{,}762/161{,}270 = \boxed{0.5070}$.

(d) Final salary is $100{,}000(1.03^{15}) = 155{,}797$. The annual payment under a monthly annuity-due is

$$25\big(0.015(100{,}000) + 0.02(55{,}797)\big) = 65{,}398$$

By Woolhouse's formula to two terms, $\ddot{a}_{65}^{(12)} = \ddot{a}_{65} - \frac{11}{24}$, so $\ddot{a}_{65} = 11\frac{11}{24}$, and

$$63{,}130\ddot{a}_{65}^{(12)} = x\ddot{a}_{65}$$

$$x = 65{,}398\left(\frac{11}{11\frac{11}{24}}\right) = \boxed{62{,}782}$$

(e) This change only affects Liu. We must recalculate ${}_{30}E_{35}$ for Liu. We'll calculate it from first principles, although you may also calculate ${}_{5}E_{35}$ and then multiply by ${}_{25}E_{40}$ which can be calculated from the pure endowment columns of the Illustrative Life Table.

$${}_{30}p_{35} = {}_{4}p_{35}\, p_{39}\, {}_{25}p_{40}$$

$$= \left(\frac{9{,}337{,}427}{9{,}420{,}657}\right)(1 - 0.00244)\left(\frac{7{,}533{,}964}{9{,}313{,}166}\right) = 0.799855$$

$${}_{30}E_{35} = \frac{0.799855}{1.06^{30}} = 0.13926$$

The revised liability for Liu is

$$5\big((0.015)(50{,}000)\big)(0.13926)(11) = 5744.6$$

instead of the previous 5743.6 calculated in part (a). The actuarial liability increases by 1 and becomes **267,202**.